Peer group situations and games

Rodica Branzei
Faculty of Computer Science, Alexandru Ioan Cuza University, Iaşi, Romania

based on

- **Tree-connected peer group situations and peer group games, R. Branzei, V. Fragnelli and S. Tijs, Mathematical Methods of Operations Research 55 (2002), 93-106**
- **Peer group situations with interval uncertainty and related games, R. Branzei, L. Mallozzi and S. Tijs, preprint no. 64, Dipartimento di Matematica e Applicazioni, Universita' di Napoli (2008)**

Outline

Part I

Preliminaries on cooperative TU games

- $< N, v >$, $N := \{1, 2, ..., n\}$: set of players
- $v : 2^N \rightarrow \mathbb{R}$: characteristic function, $v(\emptyset) = 0$
- $v(S)$: worth (or value) of coalition S

G^N: the class of all coalitional games with player set N
A game $< N, v >$ is called *superadditive* if
$v(S \cup T) \geq v(S) + v(T)$ for all $S, T \in 2^N$ with $S \cap T = \emptyset$.
A game $v \in G^N$ is called *convex* if and only if
$v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for each $S, T \in 2^N$.
$v \in G^N$ is monotonic, i.e., $v(S) \leq v(T)$ if for each $S, T \in 2^N$ with $S \subset T$.

Preliminaries on cooperative *TU* games

$v \in G^N$ is zero-normalized if $v(\{i\}) = 0$ for each $i \in N$.
$v \in G^N$ is veto-rich if $\exists k \in N$, $v(S) = 0$ if $k \notin S$: k-veto player.

The unanimity games u_T (or $< N, u_T >$), $T \in 2^N \setminus \{\emptyset\}$, which are defined by

$$u_T(S) := \begin{cases} 1, & \text{if } T \subset S \\ 0, & \text{otherwise.} \end{cases}$$

$v \in G^N$ is T-component additive game if it is superadditive, zero-normalized and $R_T(v) = v$, where $R_T(v)(S) = \sum_{u \in S \setminus T} v(u)$. Here $S \setminus T$ stands for the set of connected components of T.

Preliminaries on cooperative *TU* games

The core (Gillies (1959)) of a game $< N, v >$ is the set

$$C(v) := \left\{ x \in I(v) | \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in 2^N \setminus \{\emptyset\} \right\}.$$

Let $\Pi(N)$ be the set of all permutations $\sigma : N \to N$ of N.
The set $P^\sigma(i) := \left\{ r \in N | \sigma^{-1}(r) < \sigma^{-1}(i) \right\}$ consists of all predecessors of i with respect to the permutation σ.
Let $v \in G^N$ and $\sigma \in \Pi(N)$. The *marginal contribution vector* $m^\sigma(v) \in \mathbb{R}^n$ with respect to σ and v has the i-th coordinate the value $m_i^\sigma(v) := v(P^\sigma(i) \cup \{i\}) - v(P^\sigma(i))$ for each $i \in N$.

Preliminaries on cooperative *TU* games

The *Shapley value* (Shapley (1953)) $\phi(v)$ of a game $v \in G^N$ is the average of the marginal vectors of the game, i.e.,
$\phi(v) := \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(v)$.

Weber set: $W(v) = conv\left\{m^\sigma(v) | \sigma \in \Pi(N)\right\}$

The selectope:

$$S(v) = conv\left\{m^\beta(v) \in \mathbb{R}^n | \beta : 2^N \setminus \{\emptyset\} \to N, \beta(S) \in S\right\},$$

where $m^\beta(v)$ is the selector value corresponding to β.

Peer group situations

In many economic and OR situations the social configuration of the organization influences the potential possibilities of all the groups of agents:

- the set of agents is (strictly) hierarchically structured with a unique leader.
- the potential individual economic possibilities interfere with the behavioristical rules induced by the organization structure.

Peer group situations

In a strict hierarchy every agent has a relationship with the leader either directly or indirectly with the help of one or more other agents	$\Rightarrow$

The important group for an agent consists of the leader, the agent himself and all the intermediate agents that exist in the given hierarchy (*peer group*)

The hierarchy may be described by a rooted directed tree with the leader in the root, each other agent in a distinct node and the peer group of each agent corresponds to the agents in the unique path connecting the agent with the leader.

Peer group games

To each tree-connected peer group situation it is possible to associate a *peer group game*, with the agents as players and the characteristic function defined by pooling the individual economic possibilities of those members whose peer groups belong to the coalition.

Peer groups are essentially the only coalitions that can generate a non-zero payoff in a peer group game.

Peer group games

Peer group games form a cone, generated by unanimity games associated to peer groups, that lies in the intersection of the cones of convex games and monotonic veto-rich games (Arin, Feltkamp (1997)) with the leader as veto-player.
Related classes:

- Games with communication structures - Myerson (1977, 1980), Owen (1986), Borm, van den Nouweland, Tijs (1994)
- Trading games - Deng, Papadimitriou (1994), Topkis (1998)
- Games with permission structures - Gilles, Owen, van der Brink (1992)

Peer group games

Notations

Given a set of agents $N = \{1, 2, ..., n\}$ with a strict hierarchy, let T the associated tree.

A *T-connected peer group situation* is a triplet $< N, P, a >$ where $P : N \longrightarrow 2^N$ is a mapping which associates to each agent $i \in N$ the *T-connected peer group of agent i*, $P(i) = [1, i]$, and $a \in \mathbb{R}_+^N$ is the vector of potential economic possibilities of each agent if all his superiors cooperate with him.

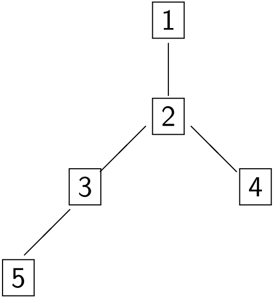
The associated *peer group game* is $\langle N, v_{P,a} \rangle$, or $< N, v >$, with

$$v(S) = \sum_{i:P(i) \subset S} a_i, \ \forall S \subset N; \ \ v(\emptyset) = 0$$

If $1 \notin S$ then $v(S) = 0$.

Example 1.

Given a set of agents $N = \{1, 2, 3, 4, 5\}$ with associated tree T:



The set of all the peer groups is:

$$[1,1] = \{1\},\ [1,2] = \{1,2\},\ [1,3] = \{1,2,3\},\ [1,4] = \{1,2,4\},$$
$$[1,5] = \{1,2,3,5\}$$

Peer group games

The peer group game is:

$$
\begin{aligned}
&v(1) = v(1,3) = v(1,4) = v(1,5) = v(1,3,4) = \\
&v(1,3,5) = v(1,4,5) = v(1,3,4,5) = a_1 \\
&v(1,2) = v(1,2,5) = a_1 + a_2 \\
&v(1,2,3) = a_1 + a_2 + a_3 \\
&v(1,2,4) = v(1,2,4,5) = a_1 + a_2 + a_4 \\
&v(1,2,3,4) = a_1 + a_2 + a_3 + a_4 \\
&v(1,2,3,5) = a_1 + a_2 + a_3 + a_5 \\
&v(N) = a_1 + a_2 + a_3 + a_4 + a_5 \\
&v(S) = 0 \text{ otherwise.}
\end{aligned}
$$

Main theoretical results

Each peer group game v can be expressed as:

$$v = \sum_{i=1}^{n} a_i u_{[1,i]}$$

where a_i represents the Harsanyi dividend of the peer group $[1, i]$.
Given N and a peer group structure P, peer group games form a cone $\{\langle N, v_{P,a}\rangle \mid a \in \mathbb{R}_+^N\}$, generated by the independent subset $\{u_{[1,i]} \mid i \in N\}$.
Peer group games are related to convex games, monotonic games, veto-rich games and Γ-component additive games (Potters, Reijnierse, 1995).

Main theoretical results

- Peer group games are nonnegative combinations of convex games and are monotonic; agent 1 (the leader) is a veto player because $v(S) = 0$ for each $S \subset N$ with $1 \notin S$.
  So the cone of peer group games is a subcone both in the cone of convex games and in the cone of monotonic games with 1 as veto player.
- From the convexity property it follows that peer group games are superadditive.
- $w = v - a_1 u_{[1,1]}$ is zero-normalized and superadditive and is an element in the cone of T-component additive games.

Main theoretical results

Peer group games have the following properties:

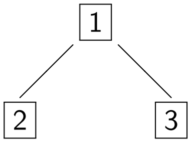
(i) *The cone of peer group games is in the intersection of the cones of convex games and monotonic veto rich games with* 1 *as veto player;*

(ii) *For each peer group game* v, *the zero-normalization* $v - a_1 u_{[1,1]}$ *is an element in the cone of* T*-component additive games.*

These properties are not characterizing for peer group games.

Example 2

Consider a T-component additive game with associated tree T:



given by

$$v = 5u_{\{1,2\}} + 7u_{\{1,3\}} + 10u_{\{1,2,3\}}.$$

v is a convex game, a T-component additive game, and also a monotonic game with 1 as veto player, but it is not a peer group game because $\{1,2,3\}$ is not a peer group.

1.Auctions situations and peer group games

In a sealed bid second price auction the seller has a reservation price r, which is known to n potential bidders (players) $1, 2, ..., n$, each of them submitting one bid $b_1, b_2, ..., b_n$.

The bidder with the highest bid obtains the object at the price of the second highest bid.

The value w_i of the object for player i, with $w_1 > w_2 > w_3 > \cdots > w_n \geq r$ is not necessarily known by the other players.

For player i a dominant strategy is $b_i = w_i \Rightarrow v(1) = w_1 - w_2, v(i) = 0$ if $i \neq 1$.

For N a dominant strategy is $b_1 = w_1$ and $b_i = r$ if $i \neq 1 \Rightarrow v(N) = w_1 - r$.

For $S \neq N$ a dominant strategy is $b_{i(S)} = w_{i(S)}$ and $b_i = r$ if $i \neq i(S) \Rightarrow v(S) = 0$ if $1 \notin S$ or $v(S) = w_1 - w_{k+1}$ if $[1, k] \subset S$ and $k+1 \notin S$ ($i(S)$ is the player with the highest bid in S).

1.Auctions situations and peer group games

Proposition 1. *$\langle N, v\rangle$ coincides with the peer group game corresponding to the T-connected peer group situation $< N, P, a >$ where $N = \{1, 2, ..., n\}$, T is a line-graph and $a_i = w_i - w_{i+1}$ for $i \in N$ with $w_{n+1} = r$.*

Example 3. In a sealed bid second price auction there are three bidders with $w = (100, 80, 50)$ and $r = 25$. The corresponding game is $v = 20u_1 + 30u_{1,2} + 25u_{1,2,3}$.

1.Auctions situations and peer group games

In a first price sealed bid auction for which $w_1, w_2, ..., w_n$ and $r = w_{n+1}$ are common knowledge among the agents, let ε be the minimal increment, with $\varepsilon < w_i - w_{i+1}, \ \forall i \in N$.
The optimal strategy for player i is $b_i := w_{i+1} + \varepsilon$.
For a subgroup S with $[1, i] \subset S$ and $i + 1 \notin S$ the optimal strategy is $b_1 = w_{i+1} + \varepsilon$, and $b_i = r, i \in S \setminus \{1\}$.

$\langle N, v\rangle$ is a peer group game, with

$$v = (w_1 - w_2 - \varepsilon)u_{\{1\}} + \sum_{i=2}^{n}(w_i - w_{i+1})u_{[1,i]}.$$

2.Graph-restricted binary communication situations

Consider a situation where gains a_i are made via binary interactions of a central agent 1 with each of the other agents $i \in N$, with restrictions on a tree T.
Interactions may be information exchange between 1 and i, or import (export) of goods via harbour 1 for agent i, or approval by 1 of a planned action of player i.
The communication restrictions lead to the peer group game

$$v = \sum_{i=1}^{n} a_i u_{[1,i]}.$$

3.Sequencing situations

A sequencing situation is a triplet (σ_0, p, α), where σ_0 is the initial order, p is the vector of processing times and α is the vector of the costs per unit of time.
$u_i = p_i^{-1}\alpha_i, i \in N$ is the urgency index of agent i.

It is optimal to serve the agents according to their urgency (Smith, 1956); this order can be obtained by neighbour switches.

3.Sequencing situations

The corresponding sequencing game (Curiel, Pederzoli, Tijs, 1989) is a nonnegative combination of unanimity games on neighbours that switch

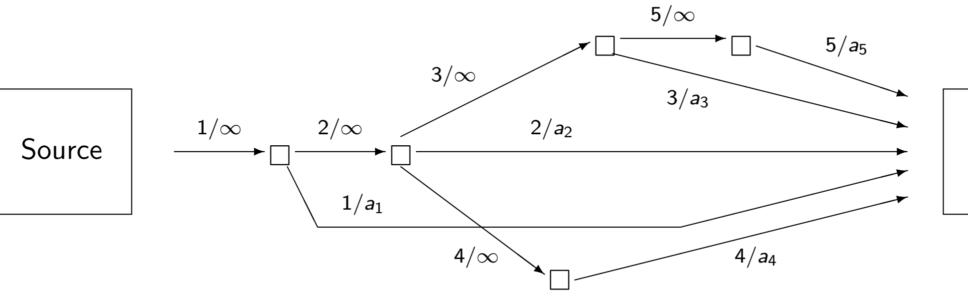
$$v = \sum_{(k,\ell),k<\ell} g_{k,\ell} u_{[k,\ell]},$$

where $g_{k,\ell}=(p_k\alpha_\ell - p_\ell\alpha_k)_+$.
If the initial order σ_0 is such that $u_2 > u_3 > \cdots > u_n$, as $g_{k,\ell} = 0$ if $k \neq 1$, it results in the peer group game:

$$v = \sum_{i=2}^{n} g_{1,i} u_{[1,i]}.$$

4.Flow situations

Given the following flow situation, where the notation is owner/capacity:



The corresponding flow game (Kalai, Zemel, 1982) is equal to the peer group game given in Example 1.

5.Airport situations

Planes of players $1, 2, ..., n$ need landing strips of length $\ell_1, \ell_2, ..., \ell_n$ with $\ell_1 > \ell_2 > \cdots > \ell_n$ and related costs $c_1 > c_2 > \cdots > c_n$.
The corresponding airport game (Littlechild, Owen, 1977) $\langle N, c\rangle$ is:

$$c = c_n u^*_N + (c_{n-1} - c_n)u^*_{N\setminus\{n\}} + \cdots + (c_1 - c_2)u^*_{\{1\}},$$

where $\langle N, u^*_S\rangle$ is a game with $u^*_S(T) = 1$ if $S \cap T \neq \emptyset$ and $u^*_S(T) = 0$ otherwise.
The dual game $\langle N, c^*\rangle$ corresponding to $\langle N, c\rangle$ is $c^*(S) = c(N) - c(N \setminus S), \forall S \subset N$, can be written as

$$c^* = c_n u_N + \sum_{i=1}^{n-1}(c_i - c_{i+1})u_{\{1,2,...,i\}},$$

and is a peer group game associated to a line-graph.

Proposition 2. *For peer group games the following properties of solution concepts hold:*

(i) *The bargaining set $\mathcal{M}(v)$ coincides with the core $C(v)$; [convexity]*

(ii) *The kernel $\mathcal{K}(v)$ coincides with the pre-kernel $\mathcal{K}^*(v)$ and the pre-kernel consists of a unique point which is the nucleolus of the game; [convexity]*

(iii) *The nucleolus $Nu(v)$ occupies a central position in the core and is the unique point satisfying*

$$Nu(v) = \{x \in C(v) \mid s_{ij}(x) = s_{ji}(x),\ \forall i, j\}$$

where $s_{ij}(x) = \max\{v(S) - x(S) \mid i \in S \subset N \setminus \{j\}\}$; *[convexity]*

(iv) *The core $C(v)$ coincides with the Weber set*

$$W(v) = \mathrm{conv}\{m^{\sigma}(v) \mid \sigma \text{ permutation of } N\};$$

[Driessen (1988), Curiel (1997)].

(v) The Shapley value $\Phi(v)$ is the center of gravity of the extreme points of the core and is given by

$$\Phi_i(v) = \sum_{j:i\in P(j)} \frac{a_j}{|P(j)|},\ i \in N; \textit{[convexity, Harsanyi dividend]}$$

(vi) The τ-value is given by

$$\tau(v) = \alpha(a_1, 0, ..., 0) + (1-\alpha)(M_1(v), M_2(v), ..., M_n(v)\},$$

where $M_i(v) = \sum_{j:i\in P(j)} a_j$; [convexity]

(vii) The core $C(v)$ coincides with the selectope

$$S(v) = \mathrm{conv}\{m^\beta(v) \in \mathbb{R}^N | \beta : 2^N \setminus \{\emptyset\} \to N, \beta(S) \in S\};$$

[Derks, Haller, Peters (2000)]

(viii) *There exist population monotonic allocation schemes (pmas). [convexity]*

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Outline
Part II

Preliminaries on cooperative interval games

- $< N, w >$, N: set of players
- $w : 2^N \to I(\mathbb{R})$: characteristic function, $w(\emptyset) = [0, 0]$
- $w(S) = [\underline{w}(S), \overline{w}(S)]$: worth (value) of S

IG^N: the class of all interval games with player set N

- $< N, w >$ is supermodular if

$$w(S) + w(T) \preccurlyeq w(S \cup T) + w(S \cap T) \text{ for all } S, T \in 2^N.$$

- $< N, w >$ is convex if $w \in IG^N$ is supermodular and $|w| \in G^N$ is supermodular (or convex).

Preliminaries on cooperative interval games

$< N, w >$ is size monotonic if $< N, |w| >$ is monotonic, i.e., $|w|(S) \leq |w|(T)$ for all $S, T \in 2^N$ with $S \subset T$.

The interval imputation set:

$$\mathcal{I}(w) = \left\{ (I_1, \ldots, I_n) \in I(\mathbb{R})^N | \sum_{i \in N} I_i = w(N), I_i \succcurlyeq w(i), \forall i \in N \right\}.$$

The interval core:

$$\mathcal{C}(w) = \left\{ (I_1, \ldots, I_n) \in \mathcal{I}(w) | \sum_{i \in S} I_i \succcurlyeq w(S), \forall S \in 2^N \setminus \{\emptyset\} \right\}.$$

Preliminaries on cooperative interval games

The interval marginal vector of $w \in SMIG^N$ $m^\sigma(w)$ for player i:

$$m_i^\sigma(w) = w(P_\sigma(i) \cup \{i\}) - w(P_\sigma(i)).$$

We say that for a game $w \in T\mathcal{I}BIG^N$ a scheme

$A = (A_{iS})_{i \in S, S \in 2^N \setminus \{\emptyset\}}$ with $A_{iS} \in I(\mathbb{R})^N$ is a pmas of w if

(i) $\sum_{i \in S} A_{iS} = w(S)$ for all $S \in 2^N \setminus \{\emptyset\}$,

(ii) $A_{iS} \preccurlyeq A_{iT}$ for all $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subset T$ and for each $i \in S$.

Preliminaries on cooperative interval games

We recall the definition of unanimity interval-valued games (Branzei et al. 2008). Given $J \in \mathrm{IR}_+^n$ and $T \in 2^N \setminus \{\emptyset\}$, the unanimity game corresponding to T is defined by

$$u_{T,J}(S) = \begin{cases} \text{J}, & T \subset S, \\ [0,0], & \text{otherwise}, \end{cases}$$

for each $S \in 2^N$. We notice that $u_{T,J} = Ju_T$.

Peer group situations with interval data and related games

We consider a finite set of agents $N = \{1, 2, ..., n\}$ having a hierarchical relationship given by a directed graph $\mathcal{T}$ where each agent is located in a node in such a way that there is one node as root corresponding to agent 1 and there exists a unique directed path from each node to the root. If all the nodes lie on a single directed path we have a chain.
Now we approach peer group situations in an interval-valued games setting. Namely we suppose that each agent i does not know exactly how much he gains from the cooperation with his superiors $[1, i]$; he knows only a lower and an upper bound of his gain given by a positive real interval $A_i = [\underline{A}_i, \bar{A}_i]$.

Peer group situations with interval data and related games

In this case we call an *interval $\mathcal{T}$-connected peer group situation* or shortly *ipg-situation* any triplet $< N, P, A >$ where N, P are as in the classical case, and $A \in \mathrm{IR}_+^n$ is the vector of agents' gain intervals.

Definition 2.1 An *interval-valued peer group game* corresponding to a ipg-situation $< N, P, A >$, in short *ipg-game* is an interval-valued cooperative game $< N, w_{P,A} >$ or shortly $< N, w >$ defined by $N = \{1, 2, ..., n\}$ and w given by

$$w(S) = \sum_{i:P(i)\subseteq S} A_i, \quad \forall S \subset N, S \neq \emptyset; w(\emptyset) = [0, 0];$$

Let us note that whenever $1 \notin S$, then $w(S) = [0, 0]$.

A sequencing production situation with interval data

Example 2.1 (**Sequential Production Situations**). We consider a production situation with n departments involved in the working process of a raw material. There is a hierarchy between them: the material is processed at stage k only after the processes in stages 1,..., $k-1$. At any stage there is a fixed cost necessary to process the material. Sometimes the cost may have an additional amount, for example due to a machinery accident. Let us suppose that the cost at stage k is in between $\underline{A}_k$ and $\bar{A}_k$. If we consider $N = \{1, ..., n\}$, $P : N \to 2^N$ s.t. $P(k) = [1, k]$ and $w(S) = \sum_{i:P(i)\subseteq S} A_i, \quad \forall S \subset N$, the game $< N, w_{P,A} >$ is an interval peer group game.

A sequencing production situation with interval data

Note that $w(S) = 0$ for any coalition S such that it does not contain $[1, i]$.

For example, let us consider $N = \{1, 2, 3\}$ and $A_i = [a_i, a_i],\ a_i \in \mathbb{R},\ i = 1, 3,\ A_2 = [\underline{A}_2, \bar{A}_2]$.

We have

$$w(\{1\}) = [a_1, a_1],$$

$$w(\{1, 2\}) = [a_1 + \underline{A}_2, a_1 + \bar{A}_2],$$

$$w(\{1, 2, 3\}) = [a_1 + \underline{A}_2 + a_3, a_1 + \bar{A}_2 + a_3],$$

and $w(S) = [0, 0]$ in any other case.

A sequencing production situation with interval data

Fig. 1

The uncertainty due to department 2 affects the departments that are not its superiors. In a non-cooperative setting, the production situations have been studied in Voorneveld et al. (1999).

The cone of interval-valued peer group games

Any interval-valued peer group game can be expressed in terms of unanimity interval-valued games as specified in the following.

Proposition 2.1 *Interval-valued peer group games with fixed* N, P

form a cone

$$C^{ipg} = \{< N, w_{P,A} >, \ A \in \mathrm{IR}_{+}^{n}\}.$$

Moreover for any $< N, P, w_{P,A} > \in C^{ipg}$ *we have*

$$w_{P,A} = \sum_{i=1}^{n} u_{[1,i],A_i}.$$

Proof. For any $\alpha, \beta \in \mathbb{R}_+$ and $A, B \in \mathrm{IR}_+^n$, given the hierarchy $\mathcal{T}$ described by P, we have

$$\alpha w_{P,A} + \beta w_{P,B} = w_{P,\alpha A + \beta B}.$$

In fact for any $S \in 2^N$ we have

$$\alpha \sum_{i:P(i)\subseteq S} A_i + \beta \sum_{i:P(i)\subseteq S} B_i = \sum_{i:P(i)\subseteq S} (\alpha A_i + \beta B_i).$$

Moreover for each $S \in 2^N$ by using the definition of unanimity interval-valued games we have

$$w_{P,A}(S) = \sum_{i=1}^{n} u_{[1,i],A_i}(S). \qquad \square$$

Properties of solution concepts for such games

Proposition 2.2 *Let* $< N, P, w_{P,A} >$, $A \in \mathrm{IR}_+^n$ *be an interval-valued peer group game. Then:*

1. $< N, P, w_{P,A} >$ *is monotonic;*

2. $< N, P, w_{P,A} >$ *is convex;*

3. the interval Shapley value is given by

$\Phi_i(w) = \sum_{j:i\in P(j)} \frac{A_i}{|P(i)|}, \in N$, *where* $|P(i)|$ *is the number of elements in* $P(i)$*;*

4. the interval Weber set is contained in the interval core:
$\emptyset \neq \mathcal{W}(w) \subset \mathcal{C}(w)$, *where* $\mathcal{W}(w) = conv\{m^\sigma(w)\ s.t.\ \sigma \in \Pi(N)\}$, $m^\sigma(w)$ *being the interval marginal vector of* w *with respect to the permutation* σ*;*

5. there exists an interval population monotonic allocation scheme.

Proof. 1. Let us consider $S, T \in 2^N$ with $S \subset T$. For any $i \in N$ such that $[1, i] \subset S \subset T$ we have $u_{[1,i],A_i}(S) = A_i = u_{[1,i],A_i}(T)$, and for any $i \in N$ such that $[1, i] \subset T$ and $[1, i]$ not subset of S we have $u_{[1,i],A_i}(S) = [0, 0] \preccurlyeq u_{[1,i],A_i}(T) = A_i$. By considering the sum for $i = 1, ..., n$ we have $w_{P,A}(S) \preccurlyeq w_{P,A}(T)$.
2. As remarked in Section 3 of Branzei et al. (2008), unanimity interval games are convex interval games. By using Proposition 2.1 we have the result.

Properties of solution concepts for such games

3. By using Proposition 4.3 in Branzei et al. (2008) the interval Shapley value is the interval $\Phi_i(w) = [\Phi_i(\underline{w}), \Phi_i(\bar{w})]$; from Proposition 1 in Branzei et al (2002) the Shapley value of a peer group game $< N, P, v_{P,a} >$, $a \in \mathbb{R}^n_+$ is the vector

$$\Phi_i(v) = \sum_{j:i \in P(j)} \frac{a_i}{|P(i)|}, \; i \in N.$$

4. By using Theorem 4.1 and Proposition 4.1 in Branzei et al. (2008).

5. By using Theorem 4.2 in Branzei et al. (2008). □

Sealed bid second price auctions with uncertainty

Now, we suppose that the bid w_i of player i is not exactly determined: the bidder submit a value in an interval $W_i = [w_i - \delta_i, w_i + \delta_i]$ where $\delta_i, i = 1, ..., n$ represents bidder ith's uncertainty.

Here the bidder ith's value will be any element in $W_i = [w_i - \delta_i, w_i + \delta_i]$. These value bounds are not known to the other players.

In our model we suppose that the bidder with the highest bid obtains the object at the highest price of the second highest bid with respect to the better than operator in $\mathbb{IR}$.

Sealed bid second price auctions with uncertainty

Suppose that for the given $\delta_1, ..., \delta_n$, the following assumptions hold:

$$W_1 \succcurlyeq W_2 \succcurlyeq W_3 \succcurlyeq .....W_n \succcurlyeq [r, r], \qquad (1a)$$

$$|W_1| \succ |W_2| \succ |W_3| \succ .....|W_n| \succcurlyeq r. \qquad (1b)$$

Note that $W_n = [\underline{w}_n, \bar{w}_n] \succcurlyeq [r, r]$ means $\underline{w}_n \geq r$.
Such δ_i exist. For example, if we consider

$$\delta_i = \frac{w_i - r}{n}, i = 1, ..., n$$

we have

$$w_{i+1} - \delta_{i+1} \leq w_i - \delta_i \Leftrightarrow (w_i - w_{i+1})(1 - \frac{1}{n}) \geq 0$$

and also

$$w_{i+1} + \delta_{i+1} \leq w_i + \delta_i \Leftrightarrow (w_{i+1} - w_i)(1 + \frac{1}{n}) \leq 0$$

so that $W_i \succcurlyeq W_{i+1}$ for all $i = 1, ..., n-1$, implying that assumption $(1a)$ holds. Moreover,

$$2\frac{w_{i+1} - r}{n} \leq 2\frac{w_i - r}{n} \Leftrightarrow w_{i+1} \leq w_i$$

so that $|W_i| > |W_{i+1}|$ for all $i = 1, ..., n-1$, implying that assumption $(1b)$ holds.

Sealed bid second price auctions with uncertainty

For any possible coalition $S \subseteq \{1, 2, ..., n\}$, let us define the payoff as follows.

i) $S = \{i\}$: player i bids W_i that gives to him a payoff $w(\{i\}) = [0, 0]$ if $i \neq 1$, $w(\{1\}) = W_1 - W_2$ because player 1 obtains the object at price in W_2;

ii) $S = N$, all the players cooperate: optimal bid for player 1 is to choose W_1, for the others is to choose $[r, r]$; player 1 obtains the object at price r and the payoff to N is $w(N) = W_1 - [r, r]$;

iii) $S \subset N$, let $i(S)$ be the player with the highest (interval) value with respect to the better than operator:

- If $1 \notin S$ then $i(S)$ bids $W_{i(S)}$, the others bid r. The object goes to player 1 and the value of coalition S is $w(S) = [0, 0]$;
- If $1 \in S$ then the highest bid is W_1 and the second highest is W_{k+1} if $[1, k] \subset S$ and $k + 1 \notin S$, since $W_i = [r, r]$ for $i = 2, ..., k$. In this case the value of the coalition is $w(S) = W_1 - W_{k+1}$.

Sealed bid second price auctions with uncertainty

Let us define $\mathcal{T}$ the line graph with root 1 and arcs $(i, i+1)$, $i = 1, ..., n-1$, and the hierarchy function $P : N \rightarrow 2^N$, $P(i) = \{1, ..., i\}$. Let us define $A_i = W_i - W_{i+1}$ for any $i \in N$: the triplet $< N, P, A >$ with $A \in \mathrm{IR}_+^n$ is a peer group situation.
The **auction peer group interval game** is the cooperative interval game $< N, w_{P,A} >$ or $< N, w >$ where

$$w_{P,A}(S) = \sum_{i=1}^{n} u_{[1,i],W_i - W_{i+1}}(S) \quad \forall S \in 2^N,$$

where $W_{n+1} = [r, r]$. In fact for any $S \subset N$ we have: $w(S) = [0, 0]$ if $1 \notin S$, $w(S) = \sum_{i=1}^{k}(W_i - W_{i+1}) = W_1 - W_{k+1}$ if $[1, k] \subset S$ and $k + 1 \notin S$.

Sealed bid second price auctions with uncertainty

Example 2.1 As in Example 4 in Branzei et al. (2002), we consider three bidders in an auction have values for the object of 100, 80, 50, respectively, and the reservation price is $r = 25$. Let us suppose that the three bidders are facing interval uncertainty with $\delta_i = \frac{w_i - r}{3}$, $i = 1, 2, 3$. We have

$$W_1 = [100 - 25, 100 + 25], \;\; W_2 = [80 - \frac{55}{3}, 80 + \frac{55}{3}]$$

$$W_3 = [50 - \frac{25}{3}, 50 + \frac{25}{3}].$$

Sealed bid second price auctions with uncertainty

Let us define the auction peer group interval game $< N, w >$ where $N = \{1, 2, 3\}$ and the characteristic function is

$$w(\{1\}) = W_1 - W_2 = [\frac{115}{3} - 25, \frac{115}{3} + 25],$$

$$w(\{1, 2\}) = W_1 - W_3 = [\frac{175}{3} - 25, \frac{175}{3} + 25],$$

$$w(\{1, 2, 3\}) = W_1 - r = [75 - 25, 75 + 25],$$

and for any other coalition $S \subseteq N$, we have $w(S) = [0, 0]$.

Flow situations with uncertainty

Let us consider G a directed network $G = (M, L)$ consisting of the set of nodes $M = \{s, 1, ..., n, t\}$ and the set of arcs $L = \{1, ..., l\}$. Each arc $l \in L$ belongs to one player with given capacity q_l and has a feasible flow $f_l \leq q_l$.

We distinguish in M a source node s and a sink node t having a very large capacity. Let S be a coalition of players, we denote by $G_{|S}$ the restricted network to only arcs owned by players in S.

Flow situations with uncertainty

We call $< M, L, f, q >$ a *flow situation*, where $f, q : L \to \mathbb{R}_+$.
With each flow situation one can associate a flow game. Recall the definition of a *flow game* (Kalai and Zemel, 1982): it is a cooperative game $< M, v >$ such that any coalition $S \subseteq M$ has value the maximum admissible flow using arcs which connect players in S with the sink node t.
Sometimes, we have only estimates about the flow of arc l, i.e. a lower bound $\underline{f}_l$ and an upper bound $\overline{f}_l$ of the real flow. So we assume that the flow of arc l is in the real interval $F_l = [\underline{f}_l, \overline{f}_l]$. In this case $F : L \to I\mathbb{R}_+$ and $< M, L, F, q >$ is called an *interval flow situation*. Here q_l represents for any $l \in L$ an upper bound of the interval F_l, i.e. $\overline{f}_l \leq q_l$.

Flow situations with uncertainty

Given an interval flow situation $< M, L, F, q >$ with $n+2$ players, let us assume that there is a hierarchical structure in the node set: a function $P : M \to M$ s.t.

$$P(1) = \{s, 1\}, \ P(i) = \{s, [1, i], t\}$$

gives the list of nodes connecting node 1 with node i, if i is connected with the sink t. Consider the case where the flow on the arcs in $P(i)$ equals a (very large) constant Q satisfying $Q > \sum_{l \in L} q_l$, except on the arc (i, t) where the admissible flow is a value in F_i.

Flow situations with uncertainty

We let:
- $N = \{1, ..., n\}$ be the set of the players;
- the hierarchy $P'(i) = [1, i] = P(i) \setminus \{s, t\}$;
- $F_i \in \mathrm{IR}_+$ be the gain of player i if players in $[1, i]$ cooperate with him.

The **flow peer group interval game** is the cooperative interval game $< N, w_{P',F} >$, or $< N, w >$, where a given coalition $S \subseteq N$ has the value

$$w(S) = \sum_{i : P'(i) \subseteq S} F_i.$$

Flow situations with uncertainty

Example 2.2 Let us consider an interval flow situation $< M, L, F, q >$ where:
- $M = \{s, 1, 2, 3, 4, 5, 6, t\}$,
- $L = \{(s,1), (1,2), (2,3), (2,4), (4,5), (1,6)\} \cup \{(i,t), i = 1,..,6\}$,
- $F_i \in \mathrm{IR}_+$ flows on arcs (i,t) for $i = 1,..,6$, and $F_l = [Q, Q]$ with $Q > \sum_{l \in L} q_l$ for $l \in \{(s,1), (1,2), (2,3), (2,4), (4,5), (1,6)\}$.

Given the hierarchy $P(i) = \{s, [1,i], t\}$, the flow peer group interval game is $< N, w >$, where $N = \{1,..,6\}$ and

$$w(S) = \sum_{i:P'(i) \subset S} F_i \text{ for any coalition } S \subseteq N.$$

For example, $w(\{5,6\}) = 0$ or $w(\{1,2,3,6\}) = F_1 + F_2 + F_3 + F_6$.

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