Draft: Life insurance mathematics in discrete time

Tom Fischer

Darmstadt University of Technology, Germany

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A recent version of the lecture notes can be downloaded under
`www.mathematik.tu-darmstadt.de/~tfischer/Ankara.pdf`

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Dr. Tom Fischer
Technische Universität Darmstadt
Fachbereich Mathematik, AG 9
Schloßgartenstr. 7
64289 Darmstadt
Germany

tfischer@mathematik.tu-darmstadt.de
www.mathematik.tu-darmstadt.de/~tfischer

About these notes

This is the preliminary (slide-form) version of the notes of a lecture at the Middle East Technical University in Ankara, Turkey, held by the author from April 12 to 16, 2004.

As the audience was quite inhomogeneous, the notes contain a brief review of discrete time financial mathematics. Some notions and results from stochastics are explained in the Appendix.

The notes contain several internet links to numerical spreadsheet examples which were developed by the author. The author does not (and cannot) guarantee for the correctness of the data supplied and the computations taking place.

Tom Fischer

Contents

1 Introduction

1.1 Life insurance mathematics?

- How will you finance your living standard after having retired?
- If you have children - who will finance their education if you and your partner die prematurely?
- How much would you pay
  - per month during your working life for the guarantee of 2000 EUR per month after you have retired?
  - today for the guarantee of 0.5 Mio. EUR paid on your death when you die within the next 20 years?
- What kind of information would you like to have before giving an answer?

"It's the economy, stupid." (Carville and Clinton, 1992)

- In 2003, the total **capital** hold by German LI-companies was

$$615,000,000,000 \text{ EUR.}$$

- In 2003, the aggregate sum of **premiums** paid to German life insurers was

$$67,000,000,000 \text{ EUR.}$$

- The aggregate sum of **benefits** was

$$75,400,000,000 \text{ EUR}$$

  where 11 billion EUR were used for reserve purposes.

- Imagine the total capital hold by LI-companies all over the world!
- How would *you* invest 67 billion EUR yearly in order to be prepared to pay off even bigger guaranteed(!) sums, later?
- First time in history, the German "Protector" had to save an LI-company from bankruptcy. - How could that happen?

Notation: *life insurance (mathematics) = LI(M)*

1.2 Preliminary remarks concerning the lecture

1.2.1 Intention

A brief introduction to life insurance mathematics in discrete time, with a focus on valuation and premium calculation which are considered in both, a

- classical framework with deterministic financial markets,

as well as in a

- modern framework with stochastic financial markets.

The emphasis lies on a rigorous stochastic modelling which easily allows to embedd the classical into the modern theory.

1.2.2 Warning

This is *not* a standard course in life insurance mathematics.

- Notation may differ from standard textbooks (e.g. Gerber, 1997) or papers (e.g. Møller and Norberg).
- We will not use expressions of $\ddot{a}$-type.
- Modern life insurance in *discrete*, and not continuous time - in contrast to most recent publications
- Some important topics cannot be considered, e.g. mortality statistics, enhanced premium principles or bonus theory.
- Lack of time - usually, one year courses are necessary.

1.2.3 Benefits

- Classical standard literature should easily be understood after this course.
- Modern LIM in continuous time should be better accessible, basic concepts should be clear.
- Consistent framework for classical and modern LIM in discrete time
- Basic principles of modern LIM are extensively discussed.
- Embedding of modern and classical LIM into modern financial mathematics
- “State-of-the-art” numeric examples for premium calculation and contract valuation with real data

1.3 Introductory examples

1.3.1 Valuation in classical life insurance

- 1 year time horizon, fixed interest rate r, i.e.
  1 EUR today will be worth $1+r$ EUR in 1 year,
  1 EUR in 1 year is worth $\frac{1}{1+r}$ EUR today.
- Persons $i \in \mathbb{N} \mathrel{\widehat{=}}$ independent Bernoulli variables B_i

$$B_i = \begin{cases} 1 & \text{if } i \text{ is dead after 1 year} \\ 0 & \text{if } i \text{ is alive after 1 year} \end{cases} \tag{1}$$

  $\Pr(B_i = 1) = p_1 > 0$, $\Pr(B_i = 0) = p_0 > 0$ and $p_1 + p_0 = 1$
- Life insurance contracts with payoffs $c_i B_i$, where
  $c_i \in \mathbb{R}^+$ and $0 \leq c_i \leq const$ for all i

- Classical LIM states that

$$\begin{aligned} PV^i &= (1+r)^{-1} \cdot c_i \cdot \mathbf{E}[B_i] \qquad (2) \\ &= (1+r)^{-1} \cdot c_i \cdot p_1 \\ \textbf{present value} &= \text{discounted expected payoff} \end{aligned}$$

- Reason: Present values/minimum fair prices allow hedging

$$\frac{1}{m}\sum_{i=1}^{m}((1+r)PV^i - c_iB_i) \overset{m\to\infty}{\longrightarrow} 0 \quad a.s. \qquad (3)$$

Strong Law of Large Numbers by Kolmogorov's Criterion (cf. Section 7.3)

$\Rightarrow$ Contracts are "balanced in the mean".

- The present value (2) of the contract at time 0 is also called **single net premium**.

1.3.2 Valuation in modern life insurance

- 1 year time horizon, *stochastic* financial market
- Persons $i \mathrel{\widehat{=}}$ independent Bernoullis B_i (dead $\mathrel{\widehat{=}}$ 1, alive $\mathrel{\widehat{=}}$ 0)
- Payoffs: $c_i B_i$ shares S, e.g. S $=$ 1 IBM share ("unit-linked")
- $S_0 =$ present (market) value of 1 share S at time 0
- Modern LIM states that

$$PV^i = S_0 \cdot c_i \cdot \mathbf{E}[B_i] \tag{4}$$

$\Rightarrow$ Kolmogorov's Criterion (Strong Law of Large Numbers) cannot be applied as contracts (payoff variables) can be highly dependent.

- Question: Why (4)?

2 A review of classical life insurance mathematics

2.1 Non-stochastic finance

2.1.1 The model

- Discrete finite time axis $\mathbb{T} = \{t_0, t_1, \ldots, t_n\}$, $t_0 = 0 < t_1 < \ldots < t_n$
- Deterministic financial markets

  $\Rightarrow$ Prices of securities are deterministic positive functions on $\mathbb{T}$, e.g. $S : t \mapsto (1+r)^t$
- Absence of arbitrage (No-arbitrage = NA)

  $\Rightarrow$ Riskless wins are excluded!

  $\Rightarrow$ Prices of securities are identical except for scaling (proof trivial!)

$\Rightarrow$ We can assume that there is only **one** deterministic security with price process $S = (S_t)_{t \in \mathbb{T}}$ and $S_0 = 1$ in the market.

EXAMPLE 2.1. Fixed yearly interest rate of 5%

- $\mathbb{T} = \{0, 1, 2, 3\}$ in years
- $S = (1, 1.05, 1.05^2, 1.05^3) \approx (1, 1.05, 1.1025, 1.1576)$ (compound interest)

2.1.2 The present value of a cash flow

- **Cash flow** $X_{\mathbb{T}} = (X_{t_0}, \ldots, X_{t_n}) \in \mathbb{R}^n$, i.e. at time t_i one has the deterministic payoff X_{t_i}
- Under condition (NA), the present value of the cashflow $X_{\mathbb{T}}$ at time $t \in \mathbb{T}$ is

$$PV_t(X_{\mathbb{T}}) = \sum_{k=0}^{n} \frac{S_t}{S_{t_k}} X_{t_k} = S_t \sum_{s \in \mathbb{T}} S_s^{-1} X_s. \tag{5}$$

EXAMPLE 2.2. Fixed yearly interest rate of 5%

- $\mathbb{T} = \{0, 1, 2, 3\}$ in years
- $S = (1, 1.05, 1.05^2, 1.05^3) \approx (1, 1.05, 1.1025, 1.1576)$
- $X_{\mathbb{T}} = (0, 1, 1, 1)$
- Present value of $X_{\mathbb{T}}$ at $t = 0$

$$PV_0 = 1.05^{-1} + 1.05^{-2} + 1.05^{-3} \approx 2.723$$

- Present value of $X_{\mathbb{T}}$ at $t = 3$

$$PV_3 = 1.05^2 + 1.05 + 1 = 3.1525$$

EXAMPLE 2.3 (Spreadsheet example).

`www.mathematik.tu-darmstadt.de/~tfischer/CompoundInterest.xls`

2.2 Classical valuation

2.2.1 The model

- Discrete finite time axis $\mathbb{T} = \{0, 1, \ldots, T\}$
- Deterministic financial market (cf. Subsection 2.1)
- Probability space $(B, \mathcal{B}_T, \mathbb{B})$ for the biometry

  Notation: **Biometric(al) data**- *data concerning the biological and some of the social states of human beings (e.g. health, age, sex, family status, ability to work)*
- Evolution of biometric information is modelled by a filtration of σ-algebras $(\mathcal{B}_t)_{t\in\mathbb{T}}$, i.e. $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \ldots \subset \mathcal{B}_T$
  (Information develops - to some extent - like a branching tree, an example follows below.)
- $\mathcal{B}_0 = \{\emptyset, B\}$, i.e. at $t = 0$ the state of the world is known for sure.

- **Cash flow:** $X_{\mathbb{T}} = (X_0, \ldots, X_T)$ where X_t is $\mathcal{B}_t$-measurable, i.e. a cash flow is a sequence of random payoffs
- **Example:** Claim of an insured person, e.g. $X_t = 1000$ if person died in $(t-1, t]$, $X_t = 0$ otherwise

2.2.2 The Expectation Principle

- See Section 7.4 for a short introduction to conditional expectations (including a detailed example).
- The classical present value of a t-payoff X at $s \in \mathbb{T}$ is

$$\Pi_s^t(X) := S_s \cdot \mathbf{E}[X/S_t|\mathcal{B}_s] \tag{6}$$

- The classical present value of a cashflow $X_{\mathbb{T}}$ at $s \in \mathbb{T}$ is

$$\begin{aligned} PV_s(X_{\mathbb{T}}) &:= \sum_{t \in \mathbb{T}} \Pi_s^t(X_t) \\ &= \sum_{t \in \mathbb{T}} S_s \cdot \mathbf{E}[X_t/S_t|\mathcal{B}_s] \end{aligned} \tag{7}$$

- **Reason:** The Strong Law of Large Numbers!
  A detailed justification follows later, in the general case.

EXAMPLE 2.4 (Term insurance).

- Market and time axis as in Example 2.2, i.e. $\mathbb{T} = \{0, 1, 2, 3\}$ and 5% interest per year
- $X_{\mathbb{T}} = (X_0, X_1, X_2, X_3)$
  $X_t = 1000$ if the person died in $(t-1, t]$ $(t = 1, 2, 3)$
  $X_t = 0$ else
- Mortality per year: 1%
- The single net premium

$$\begin{aligned} PV_0(X_{\mathbb{T}}) &= \sum_{t \in \mathbb{T}} \mathbf{E}[X_t/S_t] \qquad (8) \\ &= 0.01 \cdot 1000/1.05 \; + \; 0.99 \cdot 0.01 \cdot 1000/1.05^2 \\ &\qquad + 0.99^2 \cdot 0.01 \cdot 1000/1.05^3 \\ &\approx 26.97 \end{aligned}$$

Example 2.4 (continued)

- $B = \{aaa, aad, add, ddd\}$ ($a \widehat{=}$ alive, $d \widehat{=}$ dead)
- $\mathcal{B}_0 = \{\emptyset, B\}$; $\mathcal{B}_3 = \mathcal{P}(B)$
  $\mathcal{B}_1 = \{\emptyset, \{aaa, aad, add\}, \{ddd\}, B\}$
  $\mathcal{B}_2 = \{\emptyset, \{aaa, aad\}, \{add\}, \{ddd\}, \{aaa, aad, add\},$
  $\{aaa, aad, ddd\}, \{add, ddd\}, B\}$
- E.g. $\mathbb{B}(\{ddd\}) = 0.01$, $\mathbb{B}(\{aaa\}) = 0.970299$,
  $\mathbb{B}(\{aaa, aad, add\}) = 0.99$, $\mathbb{B}(\{add\}) = 0.0099$ etc.

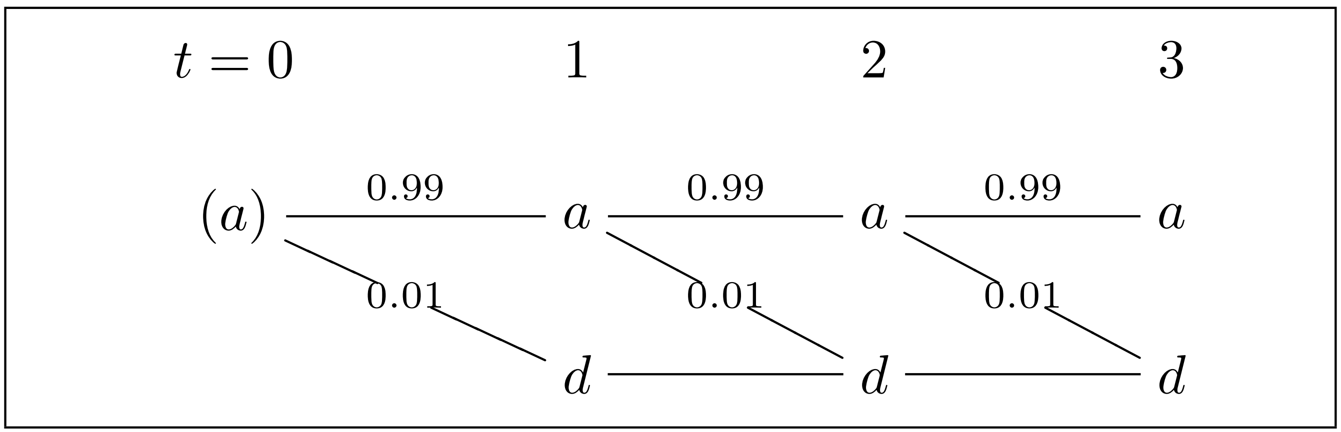


Figure 1: Stochastic tree for Example 2.4

Example 2.4 (continued)

$$PV_2(X_{\mathbb{T}}) = \sum_{t \in \mathbb{T}} S_2 \cdot \mathbf{E}[X_t/S_t|\mathcal{B}_2] = \sum_{t \in \mathbb{T}} (S_2/S_t) \cdot \mathbf{E}[X_t|\mathcal{B}_2] \tag{9}$$

First, consider

$$\mathbf{E}[X_1|\mathcal{B}_2](b) = X_1(b) = \begin{cases} 1000 & \text{if } b = ddd \\ 0 & \text{else} \end{cases} \tag{10}$$

$$\mathbf{E}[X_2|\mathcal{B}_2](b) = X_2(b) = \begin{cases} 1000 & \text{if } b = add \\ 0 & \text{else} \end{cases} \tag{11}$$

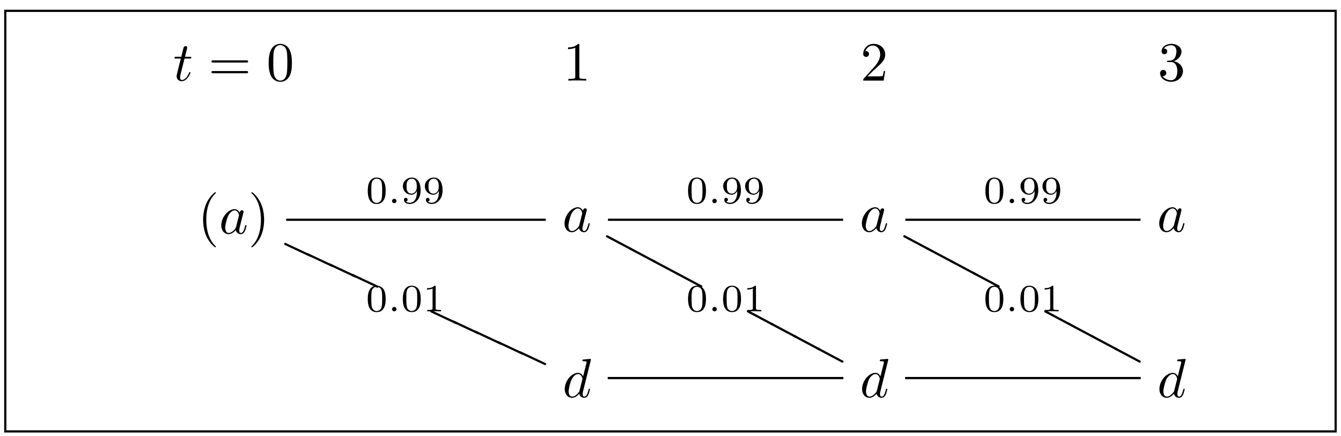
$$\mathbf{E}[X_3|\mathcal{B}_2](b) = \begin{cases} 0 & \text{if } b \in \{add, ddd\} \\ 10 & \text{else} \end{cases} \tag{12}$$

Example 2.4 (continued)

$$PV_2(X_{\mathbb{T}}) \approx \begin{cases} 1000 & \text{if } b \in \{add\} \\ 1050 & \text{if } b \in \{ddd\} \\ 9.52 & \text{else} \end{cases} \tag{13}$$

$\Rightarrow$ The contract $X_{\mathbb{T}}$ is worth/costs 9.52 EUR at time $t = 2$ if the person has not yet died.

- **Test:** $\mathbf{E}[PV_2(X_{\mathbb{T}})] \stackrel{!}{=} S_2 \cdot PV_0(X_{\mathbb{T}})$ $\Rightarrow$ OK



2.3 The fair premium

2.3.1 Life insurance contracts

- **Claims/benefits:** Cash flow $X_{\mathbb{T}} = (X_0, \ldots, X_T)$, paid from the insurer to the insurant
- **Premiums:** Cash flow $Y_{\mathbb{T}} = (Y_0, \ldots, Y_T)$, paid from the insurant to the insurer
- **Contract** from the viewpoint of the insurer: $Y_{\mathbb{T}} - X_{\mathbb{T}}$, i.e.

$$(Y_0 - X_0, \ldots, Y_T - X_T)$$

- Usually, premiums are paid in advance. Premiums can be
  - one-time single premiums.
  - periodic with constant amount.
  - periodic and varying.

2.3.2 The Principle of Equivalence

- The **net premium** or **minimum fair premium** is chosen such that

$$PV_0(X_{\mathbb{T}}) = PV_0(Y_{\mathbb{T}}), \tag{14}$$

  i.e. the present value of the premium flow has to equal the present value of the flow of benefits (*fairness* argument!).

- The Principle of Equivalence (14) and the Expectation Principle (7) are the cornerstones of classical life insurance mathematics.

- In the beginning, the value of the contract is $PV_0(Y_{\mathbb{T}} - X_{\mathbb{T}}) = 0$.

EXAMPLE 2.5 (Term insurance with constant premiums).

- Extend Example 2.4 by assuming for the premiums
  $Y_{\mathbb{T}} = (Y_0, Y_1, Y_2, Y_3)$
  $Y_t = D > 0$ if the person is alive at t $(t = 0, 1, 2)$
  $Y_t = 0$ else

- From the Principle of Equivalence we obtain

$$\begin{aligned} PV_0(Y_{\mathbb{T}}) &= D \cdot (1 + 0.99/1.05 + 0.99^2/1.05^2) \\ &\overset{!}{=} PV_0(X) \\ &\approx 26.97 \end{aligned} \tag{15}$$

- The minimum fair annual premium is $D \approx 9.52$ EUR.

2.4 Mortalities - The notation

- ${}_tq_x :=$ probability that an x year old will die within t years
- ${}_{s|t}q_x :=$ probability that an x year old will survive till s and die in $(s, s+t]$
- ${}_tp_x :=$ probability that an x year old will survive t years
- Observe: ${}_{s+t}p_x = {}_sp_x \cdot {}_tp_{x+s}$ and ${}_{s|t}q_x = {}_sp_x \cdot {}_tq_{x+s}$
- If $t = 1$, t is often omitted in the above expressions.
- **Example:** Present value of $X_{\mathbb{T}}$ from Example 2.4 in new notation $PV_0(X_{\mathbb{T}}) = 1000 \cdot (S_1^{-1} \cdot q_x + S_2^{-1} \cdot {}_{1|}q_x + S_3^{-1} \cdot {}_{2|}q_x)$
- **Caution:** Insurance companies usually use two different mortality tables depending whether a death is in financial favour (e.g. pension), or not (e.g. term insurance) for the company.

- Reason for the different tables: In actuarial practice mortality tables contain **safety loads**.
- In our examples, ${}_tq_x$, ${}_{s|t}q_x$ and ${}_tp_x$ will be taken from (or computed by) the DAV (*Deutsche Aktuarvereinigung*) mortality tables "1994 T" (Loebus, 1994) and "1994 R" (Schmithals and Schütz, 1995) *for men*.
- Hence, the used mortality tables are **first order tables**.
- The use of internal second order tables of real life insurance companies would be more appropriate. However, for competitive reasons they are usually not published.
- All probabilities mentioned above are considered to be constant in time. Especially, to make things easier, there is no "aging shift" applied to table "1994 R".

2.5 The reserve

2.5.1 Definition and meaning

- The **reserve** at time s is defined by

$$R_s(X_{\mathbb{T}}, Y_{\mathbb{T}}) := \sum_{t \geq s} \Pi_s^t (X_t - Y_t), \tag{16}$$

  i.e. (16) is the negative value of the contract cash flow after t (including t; notation may differ in the literature).

- In preparation of future payments, the company should have (16) *in reserve* at t (due to the definition, *before* contractual payments at t take place).

- Whole LI (cf. page 38): Mortality increases with time. Hence, constant premiums mean positive and increasing reserve. (The 20 year old pays the same premium as the 60 year old!)

2.5.2 A recursive formula

- By definition (16),

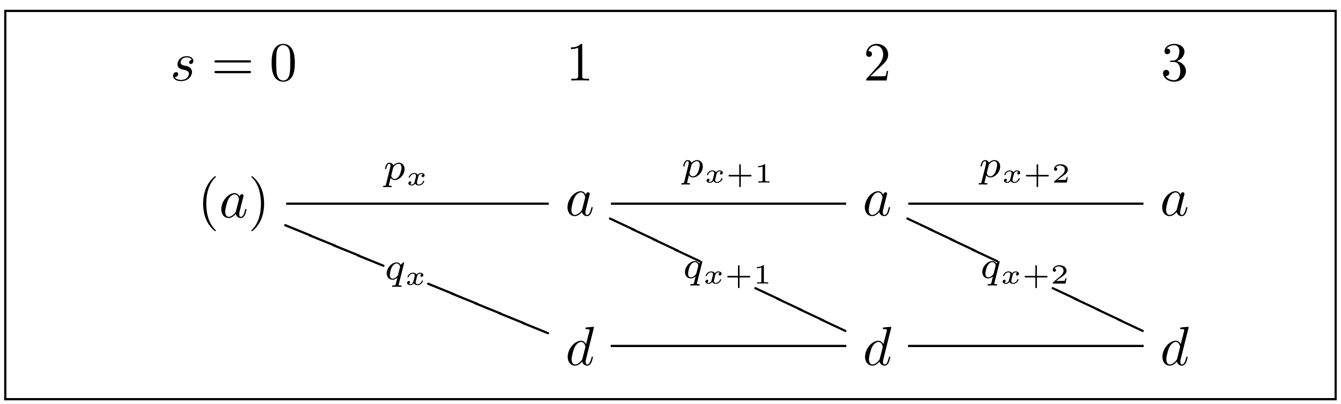
$$\begin{aligned} R_s(X_{\mathbb{T}}, Y_{\mathbb{T}}) &= \sum_{t \geq s} S_s \cdot \mathbf{E}[(X_t - Y_t)/S_t | \mathcal{B}_s] \\ &= X_s - Y_s + (S_s/S_{s+1}) \sum_{t > s} S_{s+1} \cdot \mathbf{E}[(X_t - Y_t)/S_t | \mathcal{B}_s] \\ &= X_s - Y_s + (S_s/S_{s+1}) \mathbf{E}[R_{s+1}(X_{\mathbb{T}}, Y_{\mathbb{T}}) | \mathcal{B}_s] \end{aligned} \tag{17}$$

- Note that R_s is $\mathcal{B}_s$-measurable $(s \in \mathbb{T})$.
- The reserve at t is usually considered under the assumption that the insured individual still lives.
- Assume for $(\mathcal{B}_t)_{t \in \mathbb{T}}$ the model of Example 2.4, i.e. at $t \in \mathbb{T}$ the person can be dead or alive - no other states are considered.
- We denote $R_s^a = R_s(X_{\mathbb{T}}, Y_{\mathbb{T}})|_{\{\text{alive at } s\}}$, R_s^d, X_s^a, Y_s^a analogously.

- Observe that the events $\{\text{alive at } s+1\}$ and $\{\text{alive at } s\text{, but dead at } s+1\}$ are minimal sets in $\mathcal{B}_{s+1}$.
- Hence, one obtains for an insured person of age x

$$R_s^a = (S_s/S_{s+1})[p_{x+s}R_{s+1}^a + q_{x+s}R_{s+1}^d|_{\{\text{alive at } s\}}] + X_s^a - Y_s^a. \tag{18}$$

- All expressions above are constant!
- For the contracts explained in Section 2.6, $R_{s+1}^d|_{\{\text{alive at } s\}}$ is simple to compute and (18) therefore suitable for applications (cf. Section 2.7).



2.6 Some contract forms

Payments (benefits) are normed to 1 (cf. Gerber, 1997).

- **Whole life insurance:** Provides the payment of 1 EUR at the end of the year of death. As human beings usually live not longer than 130 years, the next contract type may be used instead.
- **Term insurance of duration n:** Provides n years long the payment of 1 EUR at the end of the year of death (Example 2.4).
- **Pure endowment of duration n:** Provides the payment of 1 EUR at n if the insured is alive.
- **Endowment:** Combination of a term insurance and pure endowment with the same duration
- **Life annuity:** Provides annual payments of 1 EUR as long as the beneficiary lives (pension).

2.7 Spreadsheet examples

`www.mathematik.tu-darmstadt.de/~tfischer/ClassicalPremiums+Reserves.xls`

- Observe the differences between "1994 T" and "1994 R".
- Use different interest rates to observe how premiums depend on them.
- Implement flexible interest rates.
- Why is R_0^a, the reserve at $t = 0$, always identical to 0?

2.8 Historical remarks

Edmond Halley (1656-1742)
"An Estimate of the Degrees of the Mortality of Mankind, drawn from curious Tables of the Births and Funerals at the City of Breslaw; with an Attempt to ascertain the Price of Annuities upon Lives",
Philosophical Transactions of the Royal Society of London, 1693

- Contains the first modern mortality table.
- Proposes correctly the basics of valuation by the Expectation and the Equivalence Principle.
- Halley had a very good intuition of stochastics though not having the measure theoretical foundations of the theory.

Figure 2: Edmond Halley (1656-1742)

Pierre-Simon Marquis de Laplace (1749-1827)

- Was the first to give probability proper foundations (Laplace, 1820).
- Applied probability to insurance (Laplace, 1951).

$\Rightarrow$ Life insurance mathematics is perhaps the oldest science for which stochastic methods were developed and applied.

Figure 3: Pierre-Simon de Laplace (1749-1827)

3 Basic concepts of discrete time financial mathematics

3.1 The model

- Frictionless financial market
- Discrete finite time axis $\mathbb{T} = \{0, 1, 2, \ldots, T\}$
- $(F, (\mathcal{F}_t)_{t \in \mathbb{T}}, \mathbb{F})$ a filtered probability space, $\mathcal{F}_0 = \{\emptyset, F\}$
- Price dynamics given by an adapted $\mathbb{R}^d$-valued process $S = (S_t)_{t \in \mathbb{T}}$, i.e. d assets with price processes $(S_t^0)_{t \in \mathbb{T}}, \ldots, (S_t^{d-1})_{t \in \mathbb{T}}$ are traded at times $t \in \mathbb{T} \setminus \{0\}$.
- $(S_t^0)_{t \in \mathbb{T}}$ is called the *money account* and features $S_0^0 = 1$ and $S_t^0 > 0$ for $t \in \mathbb{T}$.
- $M^F = (F, (\mathcal{F}_t)_{t \in \mathbb{T}}, \mathbb{F}, \mathbb{T}, S)$ is called a **securities market model**.

3.2 Portfolios and strategies

- **Portfolio** due to M^F: $\theta = (\theta^0, \ldots, \theta^{d-1})$, real-valued random variables θ^i $(i = 0, \ldots, d-1)$ on $(F, \mathcal{F}_T, \mathbb{F})$

- A t-*portfolio* θ_t is $\mathcal{F}_t$-measurable. The value of θ_t at $s \geq t$

$$\langle \theta, S_s \rangle = \sum_{j=0}^{d-1} \theta^j S_s^j. \tag{19}$$

- $\mathcal{F}_t$ is interpreted as the information available at time t. Economic agents take decisions due to the available information.

$\Rightarrow$ A **trading strategy** is a vector $\theta_{\mathbb{T}} = (\theta_t)_{t \in \mathbb{T}}$ of t-portfolios θ_t.

- A **self-financing strategy** $\theta_{\mathbb{T}}$ is a strategy such that $\langle \theta_{t-1}, S_t \rangle = \langle \theta_t, S_t \rangle$ for each $t > 0$, i.e. at any time $t > 0$ the trader does not invest or consume any wealth.

3.3 No-arbitrage and the Fundamental Theorem

- S satisfies the so-called **no-arbitrage condition (NA)** if there is no s.f.-strategy such that $\langle \theta_0, S_0 \rangle = 0$ a.s., $\langle \theta_T, S_T \rangle \geq 0$ and $\mathbb{F}(\langle \theta_T, S_T \rangle > 0) > 0 \Rightarrow$ no riskless wins!
- $\overline{S} := (S_t/S_t^0)_{t \in \mathbb{T}} =$ discounted price process

THEOREM 3.1 (Dalang, Morton and Willinger, 1990). *The price process S satisfies (NA) if and only if there is a probability measure $\mathbb{Q}$ equivalent to $\mathbb{F}$ such that under $\mathbb{Q}$ the process $\overline{S}$ is a martingale. Moreover, $\mathbb{Q}$ can be found with bounded Radon-Nikodym derivative $d\mathbb{Q}/d\mathbb{F}$.*

- $\mathbb{Q}$ is called **equivalent martingale measure (EMM)**.
- 3.1 is called the Fundamental Theorem of Asset Pricing.
- The proof needs a certain form of the Hahn-Banach Theorem.

3.4 Valuation

- A **valuation principle** on a set Θ of portfolios is a linear mapping $\pi^F : \theta \mapsto (\pi_t^F(\theta))_{t\in\mathbb{T}}$, where $(\pi_t^F(\theta))_{t\in\mathbb{T}}$ is an adapted $\mathbb{R}$-valued stochastic process (price process) such that

$$\pi_t^F(\theta) = \langle \theta, S_t \rangle = \sum_{i=0}^{d-1} \theta^i S_t^i \tag{20}$$

  for any $t \in \mathbb{T}$ for which θ is $\mathcal{F}_t$-measurable.

- Fundamental Theorem $\Rightarrow$ $S' = ((S_t^0, \ldots, S_t^{d-1}, \pi_t^F(\theta)))_{t\in\mathbb{T}}$ fulfills (NA) if and only if there exists an EMM $\mathbb{Q}$ for $\overline{S'}$ - that means

$$\pi_t^F(\theta) = S_t^0 \cdot \mathbf{E}_{\mathbb{Q}}[\langle \theta, S_T \rangle / S_T^0 | \mathcal{F}_t]. \tag{21}$$

- If we price a portfolio under (NA), the price process must have the above form for some $\mathbb{Q}$. (Question: Which $\mathbb{Q}$?)

- Considering a t-portfolio θ, it is easy to show that for $s < t$

$$\pi_s^F(\theta) = S_s^0 \cdot \mathbf{E}_{\mathbb{Q}}[\langle \theta, S_t \rangle / S_t^0 | \mathcal{F}_s]. \tag{22}$$

- Sometimes it is more comfortable to use a valuation principle directly defined for payoffs.

- An $\mathcal{F}_t$-measurable payoff X always corresponds to a t-portfolio θ and vice versa. E.g. for $\theta = X/S_t^0 \cdot e_0$ one has $X = \langle \theta, S_t \rangle$, where e_i denotes the $(i-1)$-th canonic base vector in $\mathbb{R}^d$.

- So, with $X = \langle \theta, S_t \rangle$, (22) becomes

$$\Pi_s^t(X) := S_s^0 \cdot \mathbf{E}_{\mathbb{Q}}[X/S_t^0 | \mathcal{F}_s]. \tag{23}$$

- Compare (23) with (6)!

3.5 Market completeness

- A securities market model is said to be **complete** if there exists a replicating strategy for any portfolio θ, i.e. there is some self-financing $\theta_{\mathbb{T}} = (\theta_t)_{t \in \mathbb{T}}$ such that $\theta_T = \theta$.
- (NA) implies unique prices and therefore a unique EMM $\mathbb{Q}$.

THEOREM 3.2 (Harrison and Kreps (1979); Taqqu and Willinger (1987); Dalang, Morton and Willinger (1990)). *A securities market model fulfilling (NA) is complete if and only if the set of equivalent martingale measures is a singleton.*

- 3.2 is sometimes also called *Fundamental Theorem*.
- A replicating strategy is a perfect hedge.

3.6 Numeric example: One-period model

- Time axis $\mathbb{T} = \{0, 1\}$
- $F = \{\omega_1, \omega_2, \omega_3\}$, i.e. 3 states of the world at $T = 1$
- $S_0 = (1, 1, 1)$ and

	S_1^0	S_1^1	S_1^2
ω_1	1.5	1	2
ω_2	1.5	1.5	1.5
ω_3	1.5	2	1

- We want to compute the price of an option with payoff X given by $x = (X(\omega_1), X(\omega_2), X(\omega_3)) = (0.5, 1.5, 2.5)$ by two methods:
  1. by the - as we will see - uniquely determined EMM.
  2. by the respective replicating strategy/portfolio.
- For technical reasons we define the (full-rank) matrix $S_M := (S_{i,j})_{i,j=1,2,3} := (S_1^i(\omega_j))_{i,j=1,2,3}$

1.

- The condition for a EMM $\mathbb{Q}$ is $S_M \cdot (\mathbb{Q}(\omega_1), \mathbb{Q}(\omega_2), \mathbb{Q}(\omega_3))^T \cdot \frac{1}{1.5} = (1,1,1)^T$.
- The solution is $\mathbb{Q}(\omega_1) = \mathbb{Q}(\omega_2) = \mathbb{Q}(\omega_3) = 1/3$ and unique, as S_M has full rank.
- $x/S_1^0 = (1/3, 1, 5/3)$
- $\Pi_0^1(X) = \mathbf{E}_{\mathbb{Q}}[X/S_1^0] = 1/9 + 1/3 + 5/9 = 1$.

2.

- The hedge θ with $S_M^T \theta^T = x^T$ is uniquely given by $\theta = (1, 1, -1)$.
- Its price is $S_0 \theta^t = 1$.

$\Rightarrow$ As expected (NA!), both methods determine the same price.

3.7 Example: The Cox-Ross-Rubinstein model (CRR)

- 1 bond: $S_t^0 = (1+r)^t$ for $t \in \{0, 1, \dots, T\}$ and $r > 0$
- 1 stock as in Figure 4, i.e. $S_0^1 > 0$, $\mathbb{F}(S_2^1 = udS_0^1) = p^2(1-p)^2$, etc. with $0 < d < u$ and $0 < p < 1$
- Condition for EMM: $S_0^1 = \frac{1}{1+r}(p^* u S_0^1 + (1-p^*) d S_0^1)$
  $\Rightarrow p^* = \frac{1+r-d}{u-d}$
- Indeed, p^* gives a unique EMM as long as $u > 1+r > d$.

$\Rightarrow$ The CRR model is arbitrage-free and complete.

- p^* does not depend on the "real-world" probability p!

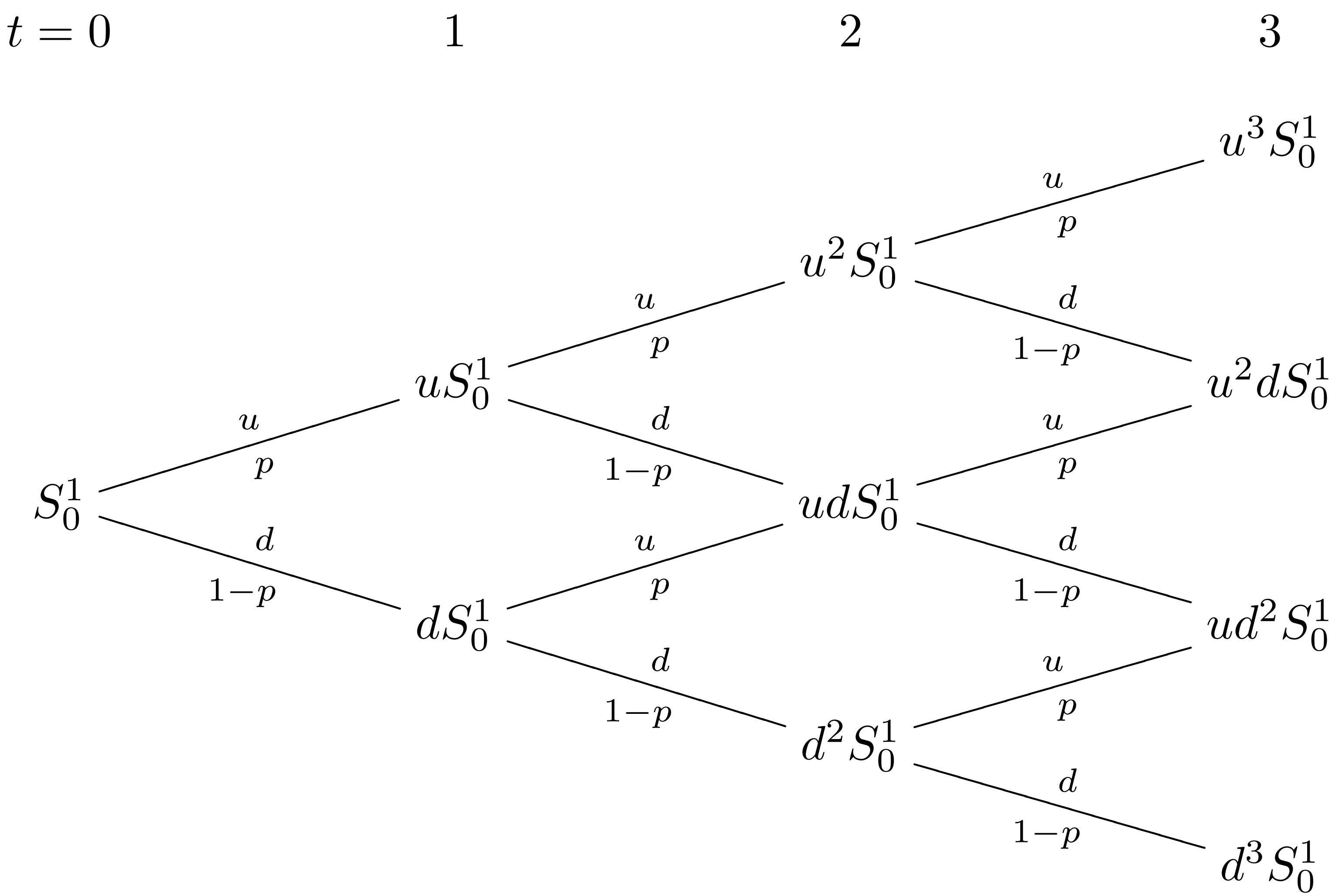


Figure 4: Binomial tree for the stock in the CRR model (T=3).

- The CRR model is complete!
- Replicating strategies for all types of options can be computed by backward induction.
- Imagine being at $t = 2$ in the state uu and having to hedge an option with payoff X at T (one-period sub-model!).

$\Rightarrow$ We simply have to solve the following equation for s and b:

$$\begin{aligned} s \cdot S_3^1(uuu) + b \cdot S_3^0(uuu) &= X(uuu) \\ s \cdot S_3^1(uud) + b \cdot S_3^0(uud) &= X(uud). \end{aligned} \tag{24}$$

s is the number of stocks, b the number of bonds.

- The hedge portfolios at other times and states can be computed analogously (going back in time).
- Observe, that (23) and the computed hedge automatically generate the same value process (if not $\Rightarrow$ arbitrage!).

3.8 Numeric example with spreadsheet

- $S_0^1 = 100$; $u = 1.06$, $d = 1.01$, $r = 0.05 \Rightarrow p^* = 0.8$
- Consider a European call option with maturity $t = 3$ and strike price $K = 110$ EUR, i.e. with value $X = (S_3^1 - 110)^+$ at $t = 3$.
- Compute the price process $(\Pi_t^T(X))_{t \in \mathbb{T}}$ of the option by Equation (23) (see Figure 5 for the solution).

`www.mathematik.tu-darmstadt.de/~tfischer/CRR.xls`

- Try to understand the spreadsheet, expecially the computing *of* and the tests *for* the replicating strategy!
- Use the spreadsheet to price and replicate an arbitrarily chosen payoff (option) at time $t = 3$.

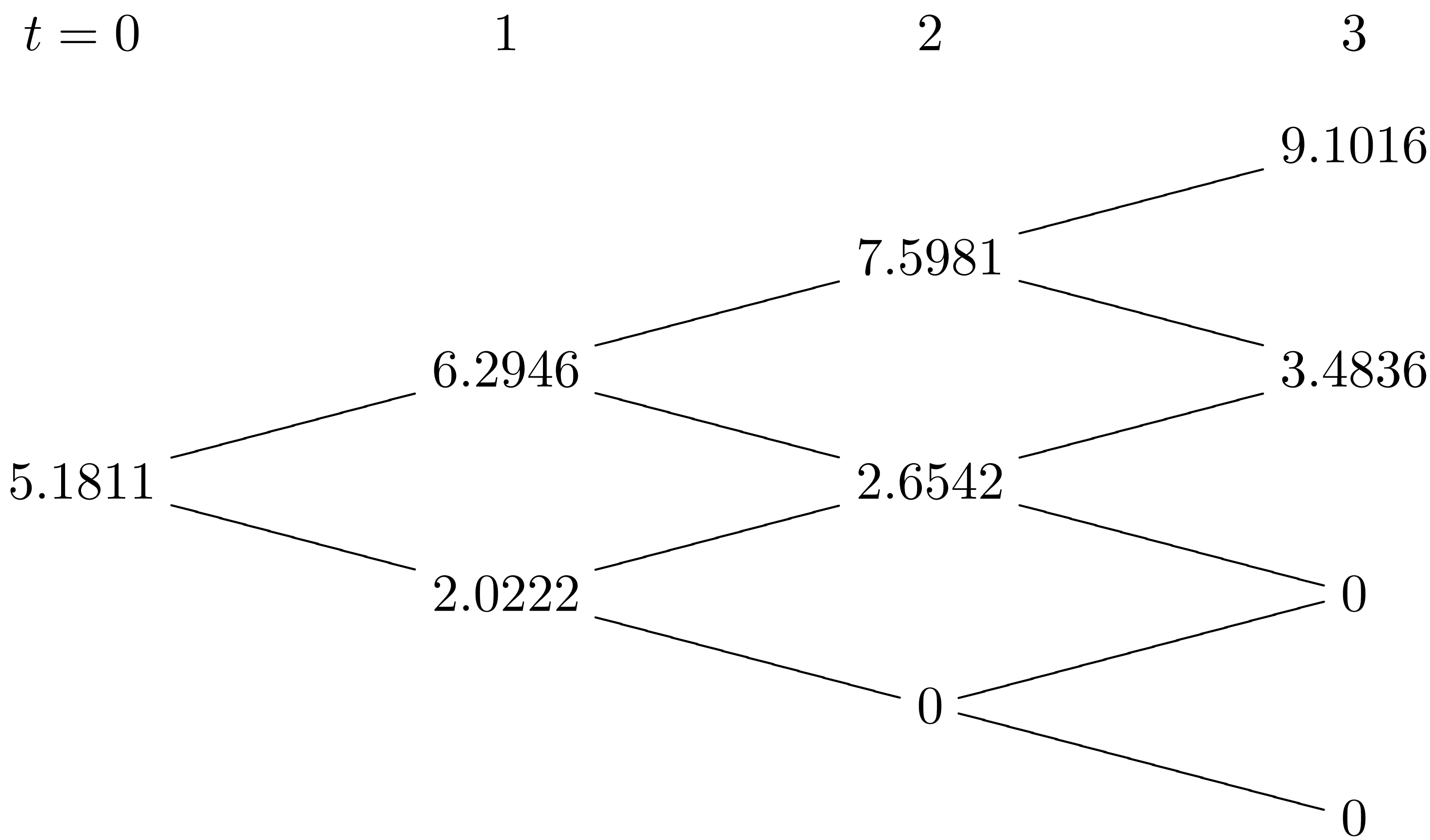


Figure 5: The price process $(\Pi_t^T(X))_{t\in\mathbb{T}}$ of the European call

4 Valuation in modern life insurance mathematics

Literature: F. (2003, 2004)

4.1 An introduction to modern valuation

- Product space $(F, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{F}) \otimes (B, (\mathcal{B}_t)_{t\in\mathbb{T}}, \mathbb{B})$; $\mathbb{T} = \{0, \ldots, T\}$
- d assets with price process(es) $S = ((S_t^0, \ldots, S_t^{d-1}))_{t\in\mathbb{T}}$
- Complete arbitrage-free financial market, unique EMM $\mathbb{Q}$
- Portfolio $\theta = (\theta^0, \ldots, \theta^{d-1})$
  (vector of integrable $\mathcal{F}_T \otimes \mathcal{B}_T$-measurable random variables)
- Random payoff $\langle \theta, S_T \rangle = \sum_{j=0}^{d-1} \theta^j S_T^j$ at time T

$$\begin{aligned} PV_0(\theta) &= \pi_0(\theta) = \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle \theta, S_T \rangle / S_T^0] \\ &\overset{\text{Fubini}}{=} \mathbf{E}_{\mathbb{Q}}[\langle \mathbf{E}_{\mathbb{B}}[\theta], S_T \rangle / S_T^0] \end{aligned} \tag{25}$$

fair/market price of biometrically expected portfolio

- History: Brennan and Schwartz (1976), Aase and Persson (1994), Persson (1998) and others

EXAMPLE 4.1 (cf. Section 1.3.2).

- 1 year time horizon ($T = 1$)
- Person modelled as a Bernoulli variable B_i (dead $\widehat{=}$ 1, alive $\widehat{=}$ 0)
- Payoff: $X = c_i B_i S_T^1 = c_i B_i$ shares of type 1 at T ("unit-linked")
- $S_0^1 =$ present (market) value of 1 share at time 0
- The present value of X

$$
\begin{aligned}
\Pi_0^T(X) &= \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[X/S_T^0] && (26)\\
&\overset{\text{Fubini}}{=} \mathbf{E}_{\mathbb{Q}}[c_i \cdot \mathbf{E}_{\mathbb{B}}[B_i] \cdot (S_T^1/S_T^0)]\\
&= c_i \cdot \mathbf{E}_{\mathbb{B}}[B_i] \cdot \mathbf{E}_{\mathbb{Q}}[S_T^1/S_T^0]\\
&= S_0^1 \cdot c_i \cdot \mathbf{E}_{\mathbb{B}}[B_i]
\end{aligned}
$$

- Reasons for the product measure approach (25)
  - risk-neutrality towards biometric risks
  - minimal martingale measure $\mathbb{Q} \otimes \mathbb{B}$
  - $\mathcal{F}_T = \{\emptyset, F\}$ implies classical Expectation Principle
- Questions

  $\Rightarrow$ Does the product measure approach follow from the demand for hedges such that mean balances converge to 0 a.s.?

  $\Rightarrow$ Can we find a system of axioms for modern life insurance?
- Result

  8 principles (7 axioms) are a reasonable framework for modern life insurance and imply the product measure approach

4.2 Principles of modern life insurance mathematics

1. Independence of biometric and financial events
   - Death or injury of persons independent of financial events (e.g. Aase and Persson, 1994)
   - Counterexamples may occur in real life
2. Complete arbitrage-free financial markets
   - Reasonable from the viewpoint of insurance
   - In real life, purely financial products are bought from banks or can be traded or replicated in the financial markets.
   - Literature: Assumption of a Black-Scholes model (cf. Aase and Persson (1994), Møller (1998))
3. Biometric states of individuals are independent
   - Standard assumption also in classical life insurance
   - Counterexamples like e.g. married couples are irrelevant.

4. Large classes of similar individuals
   - Large classes of individuals of the same age, sex and health status (companies have thousands of clients).
   - A company should be able to cope with such a large class even if all individuals have the same kind of contract.
5. Similar individuals can not be distinguished
   - Similar individuals (in the above sense) should pay the same premiums for the same types of contracts (fairness!).
   - Companies pursue e.g. the same hedges for the same kind of contracts with similar individuals.
6. No-arbitrage pricing
   - It should not be able to make riskless wins when trading with life insurance contracts (e.g. Delbaen and Haezendonck, 1989).
7. Minimum fair prices allow (purely financial) hedging such that

mean balances converge to 0 almost surely

- Compare with the examples in the Sections 1.3.1 and 1.3.2.
- Analogy to the classical case: The minimum fair price (net present value) of any contract (from the viewpoint of the insurer) should at least cover the price of a purely financial hedging strategy that lets the mean balance per contract converge to zero a.s. for an increasing number of clients.

8. Principle of Equivalence

- Future payments to the insurer (premiums) should be determined such that their present value equals the present value of the future payments to the insured (benefits).

$\Rightarrow$ The liabilities (benefits) can somehow be hedged working with the premiums.

- Cf. the classical case (14)

4.3 The model

AXIOM 1. *A common filtered probability space*

$$(M, (\mathcal{M}_t)_{t\in\mathbb{T}}, \mathbb{P}) = (F, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{F}) \otimes (B, (\mathcal{B}_t)_{t\in\mathbb{T}}, \mathbb{B}) \tag{27}$$

of financial and biometric events is given, i.e. $M = F \times B$, $\mathcal{M}_t = \mathcal{F}_t \otimes \mathcal{B}_t$ *and* $\mathbb{P} = \mathbb{F} \otimes \mathbb{B}$. *Furthermore,* $\mathcal{F}_0 = \{\emptyset, F\}$ *and* $\mathcal{B}_0 = \{\emptyset, B\}$.

- Biometry and finance are independent!
- $(B, (\mathcal{B}_t)_{t\in\mathbb{T}}, \mathbb{B})$ describes the development of the biological states of all considered human beings.
- No particular model for the development of the biometric information!
- Cf. Principle 1

AXIOM 2. *A complete securities market model*

$$M^F = (F, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{F}, \mathbb{T}, {}_F S) \tag{28}$$

with $|\mathcal{F}_T| < \infty$ *and a unique equivalent martingale measure* $\mathbb{Q}$ *are given. The common market of financial and biometric risks is denoted by*

$$M^{F\times B} = (M, (\mathcal{M}_t)_{t\in\mathbb{T}}, \mathbb{P}, \mathbb{T}, S), \tag{29}$$

where $S(f,b) = {}_F S(f)$ *for all* $(f,b) \in M$*.*

- $M^{F\times B}$ is understood as a securities market model.
- $|\mathcal{F}_T| < \infty$ as there are no discrete time financial market models which are complete and have a really infinite state space (cf. Dalang, Morton and Willinger, 1990).
- Cf. Principle 2

AXIOM 3. *There are infinitely many human individuals and we have*

$$(B, (\mathcal{B}_t)_{t\in\mathbb{T}}, \mathbb{B}) = \bigotimes_{i=1}^{\infty} (B^i, (\mathcal{B}_t^i)_{t\in\mathbb{T}}, \mathbb{B}^i), \tag{30}$$

where $B_H = \{(B^i, (\mathcal{B}_t^i)_{t\in\mathbb{T}}, \mathbb{B}^i) : i \in \mathbb{N}^+\}$ *is the set of filtered probability spaces which describe the development of the* i*-th individual* $(\mathbb{N}^+ := \mathbb{N} \setminus \{0\})$*. Each* $\mathcal{B}_0^i$ *is trivial.*

- $\mathcal{B}_0$ is also trivial, i.e. $\mathcal{B}_0 = \{\emptyset, B\}$.
- Cf. Principle 3

AXIOM 4. *For any space* $(B^i, (\mathcal{B}_t^i)_{t\in\mathbb{T}}, \mathbb{B}^i)$ *in* B_H *there are infinitely many isomorphic (identical, except for the index) ones in* B_H*.*

- Cf. Principle 4

DEFINITION 1. *A general* **life insurance contract** *is a vector* $(\gamma_t, \delta_t)_{t\in\mathbb{T}}$ *of pairs* (γ_t, δ_t) *of* t*-portfolios in* Θ $($*to shorten notation we drop the inner brackets of* $((\gamma_t, \delta_t))_{t\in\mathbb{T}})$. *For any* $t \in \mathbb{T}$, *the portfolio* γ_t *is interpreted as a payment of the insurer to the insurant* (**benefit**) *and* δ_t *as a payment of the insurant to the insurer* (**premium**), *respectively taking place at* t. *The notation* $({}^i\gamma_t, {}^i\delta_t)_{t\in\mathbb{T}}$ *means that the contract depends on the* i*-th individual's life, i.e. for all* $(f, x), (f, y) \in M$

$$({}^i\gamma_t(f, x), {}^i\delta_t(f, x))_{t\in\mathbb{T}} = ({}^i\gamma_t(f, y), {}^i\delta_t(f, y))_{t\in\mathbb{T}} \tag{31}$$

whenever $p^i(x) = p^i(y)$, p^i *being the canonical projection of* B *onto* B^i.

- Benefits at t: γ_t
- Premiums at t: δ_t

$\Rightarrow$ Viewpoint of the insurer: company gets $\delta_t - \gamma_t$ at t

AXIOM 5. *Suppose a suitable valuation principle* π *on* Θ*. For any life insurance contract* $(\gamma_t, \delta_t)_{t \in \mathbb{T}}$ *the* **Principle of Equivalence** *demands that*

$$\pi_0 \left(\sum_{t=0}^{T} \gamma_t \right) = \pi_0 \left(\sum_{t=0}^{T} \delta_t \right). \tag{32}$$

- Observe the analogy to the classical case.
- Cf. Principle 8 and the classical case (14)

AXIOM 6. *Any valuation principle π taken into consideration must for any $t \in \mathbb{T}$ and $\theta \in \Theta$ be of the form*

$$\pi_t(\theta) = S_t^0 \cdot \mathbf{E}_{\mathbb{M}}[\langle \theta, S_T \rangle / S_T^0 | \mathcal{F}_t \otimes \mathcal{B}_t] \tag{33}$$

for a probability measure $\mathbb{M} \sim \mathbb{P}$. Furthermore, one must have

$$\pi_t({}_F\theta) = \pi_t^F({}_F\theta) \tag{34}$$

$\mathbb{P}$-a.s. for any M^F-portfolio ${}_F\theta$ and all $t \in \mathbb{T}$, where π_t^F is the price operator in M^F.

- Cf. Principle 6

DEFINITION 2.

(i) *Define*

$$\Theta = (L^1(M, \mathcal{M}_T, \mathbb{P}))^d \tag{35}$$

and

$$\Theta^F = (L^0(F, \mathcal{F}_T, \mathbb{F}))^d, \tag{36}$$

where Θ^F *can be interpreted as a subset of* Θ *by the usual embedding since all* $L^p(F, \mathcal{F}_T, \mathbb{F})$ *are identical for* $p \in [0, \infty]$.

(ii) *A set* $\Theta' \subset \Theta$ *of portfolios in* $M^{F \times B}$ *is called* **independently identically distributed due to** $(B, \mathcal{B}_T, \mathbb{B})$, *abbreviated* **B-i.i.d.**, *when for almost all* $f \in F$ *the random variables* $\{\theta(f, .) : \theta \in \Theta'\}$ *are i.i.d. on* $(B, \mathcal{B}_T, \mathbb{B})$. *Under Axiom 4, such sets exist and can be countably infinite.*

(iii) *Under the Axioms 1 to 3, a set* $\Theta' \subset \Theta$ *satisfies condition* **(K)** *if for almost all* $f \in F$ *the elements of* $\{\theta(f,.) : \theta \in \Theta'\}$ *are stochastically independent on* $(B, \mathcal{B}_T, \mathbb{B})$ *and* $||\theta^j(f,.)||_2 < c(f) \in \mathbb{R}^+$ *for all* $\theta \in \Theta'$ *and all* $j \in \{0, \ldots, d-1\}$*.*

AXIOM 7. *Under the Axioms 1 - 4 and 6, a* **minimum fair price** *is a valuation principle π on Θ that must for any $\theta \in \Theta$ fulfill*

$$\pi_0(\theta) = \pi_0^F(H(\theta)) \tag{37}$$

where

$$H : \Theta \longrightarrow \Theta^F \tag{38}$$

is such that

(i) $H(\theta)$ *is a t-portfolio whenever θ is.*

(ii) $H({}^1\theta) = H({}^2\theta)$ *for B-i.i.d. portfolios ${}^1\theta$ and ${}^2\theta$.*

(iii) *for t-portfolios $\{{}^i\theta : i \in \mathbb{N}^+\}_{B-i.i.d.}$ or $\{{}^i\theta : i \in \mathbb{N}^+\}_K$ one has*

$$\frac{1}{m}\sum_{i=1}^{m}\langle {}^i\theta - H({}^i\theta), S_t\rangle \overset{m\to\infty}{\longrightarrow} 0 \quad \mathbb{P}\text{-a.s.} \tag{39}$$

- Cf. Pinciples 5 and 7
- Hedge $H(\theta)$ does not react on biometric events after $t = 0$.

4.4 The main result on valuation

THEOREM 4.2 (F., 2003). *Under the Axioms 1-4, 6 and 7, the minimum fair price π on Θ (=integrable portfolios) is uniquely determined by $\mathbb{M} = \mathbb{Q} \otimes \mathbb{B}$, i.e. for $\theta \in \Theta$ and $t \in \mathbb{T}$*

$$\pi_t(\theta) = S_t^0 \cdot \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle \theta, S_T \rangle / S_T^0 | \mathcal{F}_t \otimes \mathcal{B}_t]. \tag{40}$$

- $\mathbb{Q} \otimes \mathbb{B}$ is EMM for S in the product space
- Result/deduction by axiomatic approach is new
- The hedges (cf. Principle/Axiom 7) are (uniquely) determined by $\mathbf{E}_{\mathbb{B}}[.]$ (L^2-approximation)
- Proof of Theorem 4.2 is a little laborious (cf. F., 2003).
- (6) **is nothing but** (40) **for the special case** $\mathcal{F}_t = \{\emptyset, F\}$**, i.e. for deterministic financial markets.**

Some results needed for the proof of Theorem 4.2:

LEMMA 1. *Let* $(g_n)_{n\in\mathbb{N}}$ *and* g *be a sequence, respectively a function, in* $L^0(F \times B, \mathcal{F} \otimes \mathcal{B}, \mathbb{F} \otimes \mathbb{B})$*, i.e. the real valued measurable functions on* $F \times B$*, where* $(F \times B, \mathcal{F} \otimes \mathcal{B}, \mathbb{F} \otimes \mathbb{B})$ *is the product of two arbitrary probability spaces. Then* $g_n \to g$ $\mathbb{F} \otimes \mathbb{B}$*-a.s. if and only if* $\mathbb{F}$*-a.s.* $g_n(f, .) \to g(f, .)$ $\mathbb{B}$*-a.s.*

LEMMA 2. *Under Axiom 1 and 2, one has for any* $\theta \in \Theta$

$$H^*(\theta) := \mathbf{E}_{\mathbb{B}}[\theta] \in \Theta^F. \tag{41}$$

There is a self-financing strategy replicating $H^*(\theta)$ *and under Axiom 6*

$$\pi_t(H^*(\theta)) = S_t^0 \cdot \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle\theta, S_T\rangle / S_T^0 | \mathcal{F}_t \otimes \mathcal{B}_0] \tag{42}$$

for $t \in \mathbb{T}$*. Moreover,* H^* *fulfills properties* (i)*,* (ii) *and* (iii) *of Axiom 7.*

LEMMA 3. *Under the Axioms 1 - 4 and 6, any $H : \Theta \to \Theta^F$ fulfilling (i), (ii) and (iii) of Axiom 7 fulfills for any θ in some $\Theta_{B-i.i.d.}$*

$$\pi_t(H(\theta)) = S_t^0 \cdot \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle\theta, S_T\rangle / S_T^0 | \mathcal{F}_t \otimes \mathcal{B}_0], \quad t \in \mathbb{T}. \tag{43}$$

- There is no reasonable purely financial hedging method (i.e. a strategy not using biometric information) for the relevant portfolios with better convergence properties than (41).

Also nice to know:

LEMMA 4. *Under Axiom 1 and 2, for any $\theta \in \Theta$, any $t \in \mathbb{T}$ and for $\mathbb{M} \in \{\mathbb{F}\otimes\mathbb{B}, \mathbb{Q}\otimes\mathbb{B}\}$*

$$\mathbf{E}_{\mathbb{M}}[\langle\theta - H^*(\theta), S_t\rangle] = 0. \tag{44}$$

Proof. By Fubini's Theorem. □

4.5 Hedging and some implications

- Suppose Axiom 1 to 4
- Life insurance contracts $\{({}^i\gamma_t, {}^i\delta_t)_{t\in\mathbb{T}} : i \in \mathbb{N}^+\}$ with $\{{}^i\gamma_t : i \in \mathbb{N}^+\}_K$ and $\{{}^i\delta_t : i \in \mathbb{N}^+\}_K$ for all $t \in \mathbb{T}$
- Buy the portfolios (or strategies replicating) $\mathbf{E}_{\mathbb{B}}[{}^i\gamma_t]$ and $-\mathbf{E}_{\mathbb{B}}[{}^i\delta_t]$ for all $i \in \mathbb{N}^+$ and all $t \in \mathbb{T}$.
- *Mean total payoff* per contract at time t

$$\frac{1}{m}\sum_{i=1}^{m}\langle {}^i\delta_t - {}^i\gamma_t - \mathbf{E}_{\mathbb{B}}[{}^i\delta_t - {}^i\gamma_t], S_t\rangle \overset{m\to\infty}{\longrightarrow} 0 \quad \mathbb{F}\otimes\mathbb{B}\text{-a.s.} \quad (45)$$

- Also the mean *final* balance converges

$$\frac{1}{m}\sum_{i=1}^{m}\sum_{t=0}^{T}\langle {}^i\delta_t - {}^i\gamma_t - \mathbf{E}_{\mathbb{B}}[{}^i\delta_t - {}^i\gamma_t], S_T\rangle \overset{m\to\infty}{\longrightarrow} 0 \quad \mathbb{F}\otimes\mathbb{B}\text{-a.s.} \quad (46)$$

- Static risk management!
- This is *not* standard mean variance hedging. (cf. Bouleau and Lamberton (1989), Duffie and Richardson (1991))
- Other hedging approaches e.g. in Møller (2002)
- The Principle of Equivalence (32) applied under the minimum fair price (25):

$$\sum_{t=0}^{T} \pi_0(\mathbf{E}_{\mathbb{B}}[-{}^i\delta_t + {}^i\gamma_t]) = \sum_{t=0}^{T} \pi_0({}^i\delta_t - {}^i\gamma_t) = 0. \tag{47}$$

⇒ Under (32) and (25), a LI-company can without any costs at time 0 (!) pursue a s.f. trading strategy such that the mean balance per contract at any time t converges to zero almost surely for an increasing number of individual contracts.

- Realization would demand the precise knowledge of the **second order base** given by the Axioms 1 to 4.

Arbitrage like trading strategies

- Suppose $\{^i\theta, i \in \mathbb{N}^+\}_{B-i.i.d.}$ and sell $\{^1\theta, \dots, {}^m\theta\}$ at prices $\pi_0(^i\theta) + \epsilon$, where $\epsilon > 0$ is an additional fee and π is as in (25).
- Hedge each $^i\theta$ as above, which costs $\pi_0(^i\theta)$.

$\Rightarrow$ The balance converges as explained above, but additionally ϵ per contract was gained at $t = 0$.

- The safety load ϵ makes in the limit a deterministic money making machine out of the insurance company.

- L^2-framework, i.e. $\langle \theta_t, S_t \rangle$ of any considered t-portfolio θ_t lies in $L^2(M, \mathcal{M}_t, \mathbb{P})$.

$\Rightarrow$ $\mathbf{E}_{\mathbb{B}}[.]$ is the orthogonal projection of $L^2(M, \mathcal{M}_t, \mathbb{P})$ onto its purely financial (and closed) subspace $L^2(F, \mathcal{F}_t, \mathbb{F})$.

$\Rightarrow$ Hilbert space theory: The payoff $\langle \mathbf{E}_{\mathbb{B}}[\theta_t], S_t \rangle = \mathbf{E}_{\mathbb{B}}[\langle \theta_t, S_t \rangle]$ of the hedge $H^*(\theta_t)$ is the best L^2-approximation of the payoff $\langle \theta_t, S_t \rangle$ of the t-portfolio θ_t by a purely financial portfolio in M^F.

- **Minimal martingale measure:** $\mathbb{M} = \mathbb{Q} \otimes \mathbb{B}$ minimizes $||d\mathbb{M}/d\mathbb{P} - 1||_2$ under $\mathbf{E}_{\mathbb{B}}[d\mathbb{M}/d\mathbb{P}] = d\mathbb{Q}/d\mathbb{F}$ (implied by Axiom 6). Under some additional technical assumptions, this property is a characterization of the so-called *minimal martingale measure* in the time continuous case (cf. Schweizer (1995b), Møller (2001)).

$\Rightarrow$ $\mathbb{Q} \otimes \mathbb{B}$ can be interpreted as the EMM which lies "next" to $\mathbb{P} = \mathbb{F} \otimes \mathbb{B}$ due to the L^2-metric.

5 Examples

5.1 Preliminaries

5.1.1 Interest rates and zero-coupon bonds

- A financial product which guarantees the owner the payoff of one currency unit at time t is called a **zero-coupon bond (ZCB)** with maturity t.
- The price of a ZCB at time $s < t$ is denoted by $p(s, t - s)$ where $t - s$ is the **time to maturity** and $p(s, 0) := 1$.
- Accumulations (sums) of ZCBs are called **coupon bonds**; the price of a coupon bond is given by the sum of the prices of the respective ZCB it consists of
- "Real world" examples: Debt securities & government bonds (hopefully non-defaultable)

- **Spot (interest) rate** $R(t,\tau)$ for the time interval $[t, t+\tau]$

$$R(t,\tau) := -\frac{\log p(t,\tau)}{\tau} \tag{48}$$

- **Yield curve** at time t is the mapping with $\tau \mapsto R(t,\tau)$ for $\tau > 0$
- Spot Rates are continuously compounded. Discrete interest rates R' via $1 + R' = e^R$, i.e.

$$e^{-\tau R} = p = (1+R')^{-\tau} \tag{49}$$

- For any ZCB one has a corresponding interest rate R (R') and vice versa
- In a stochastic market, $(R(t,\tau))_{t\in\mathbb{T}}$ and $(p(t,\tau))_{t\in\mathbb{T}}$ are stochastic processes!

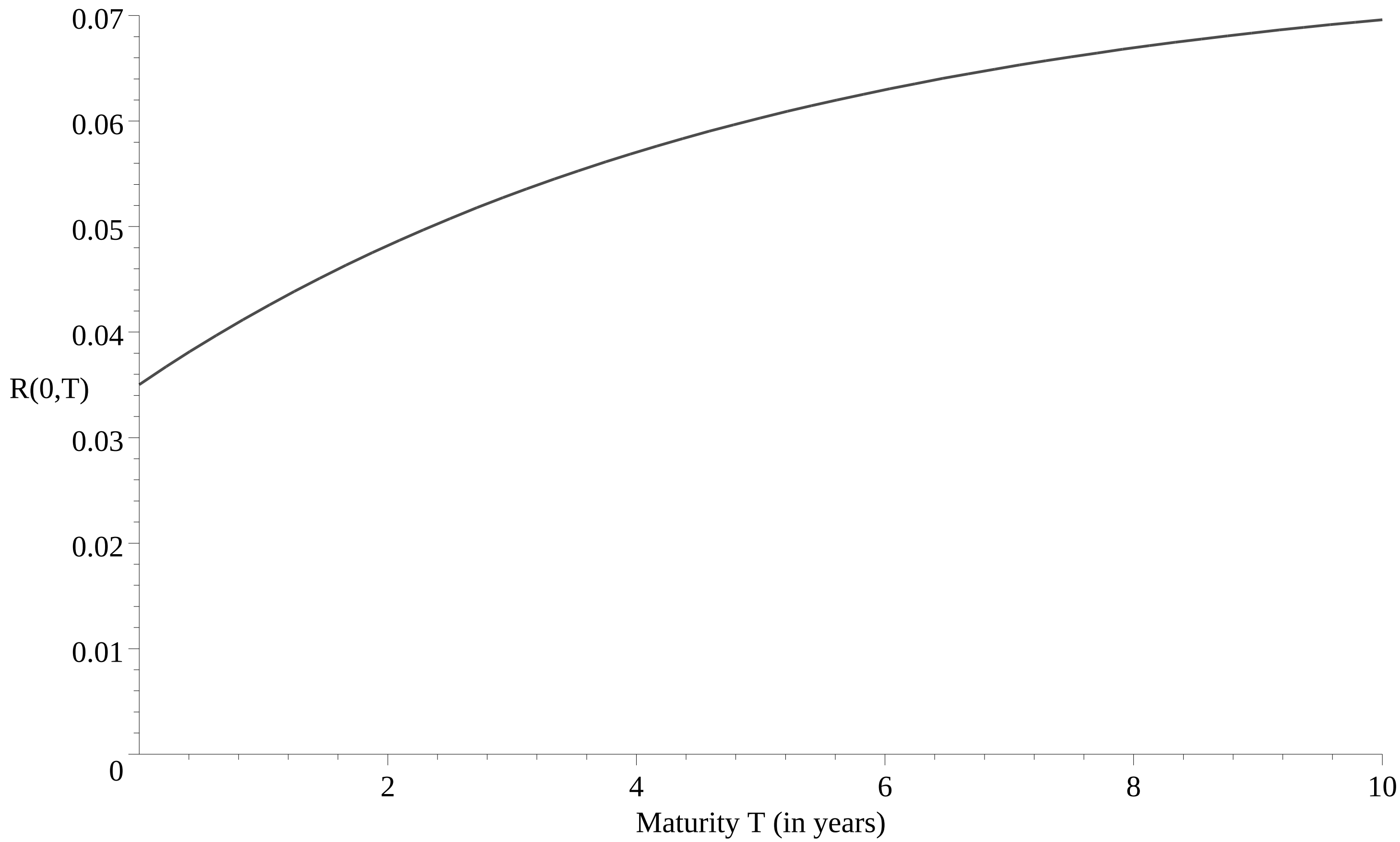


Figure 6: Hypothetical yield curve at time 0

5.1.2 Real data

- Figure 7 shows the historical yield structure (i.e. the set of yield curves) of the German debt securities market from September 1972 to April 2003 (taken from the end of each month).

- The maturities' range is 0 to 28 years. The values for $\tau > 0$ were computed via a parametric presentation of yield curves (the so-called Svensson-method; cf. Schich (1997)) for which the parameters can be taken from the Internet page of the German Federal Reserve (*Deutsche Bundesbank*; `www.bundesbank.de`).

- The implied Bundesbank values R' are estimates of *discrete* interest rates on notional zero-coupon bonds based on German Federal bonds and treasuries (cf. Schich, 1997) and have to be converted to continuously compounded interest rates (as implicitly used in (48)) by $R = \ln(1 + R')$.

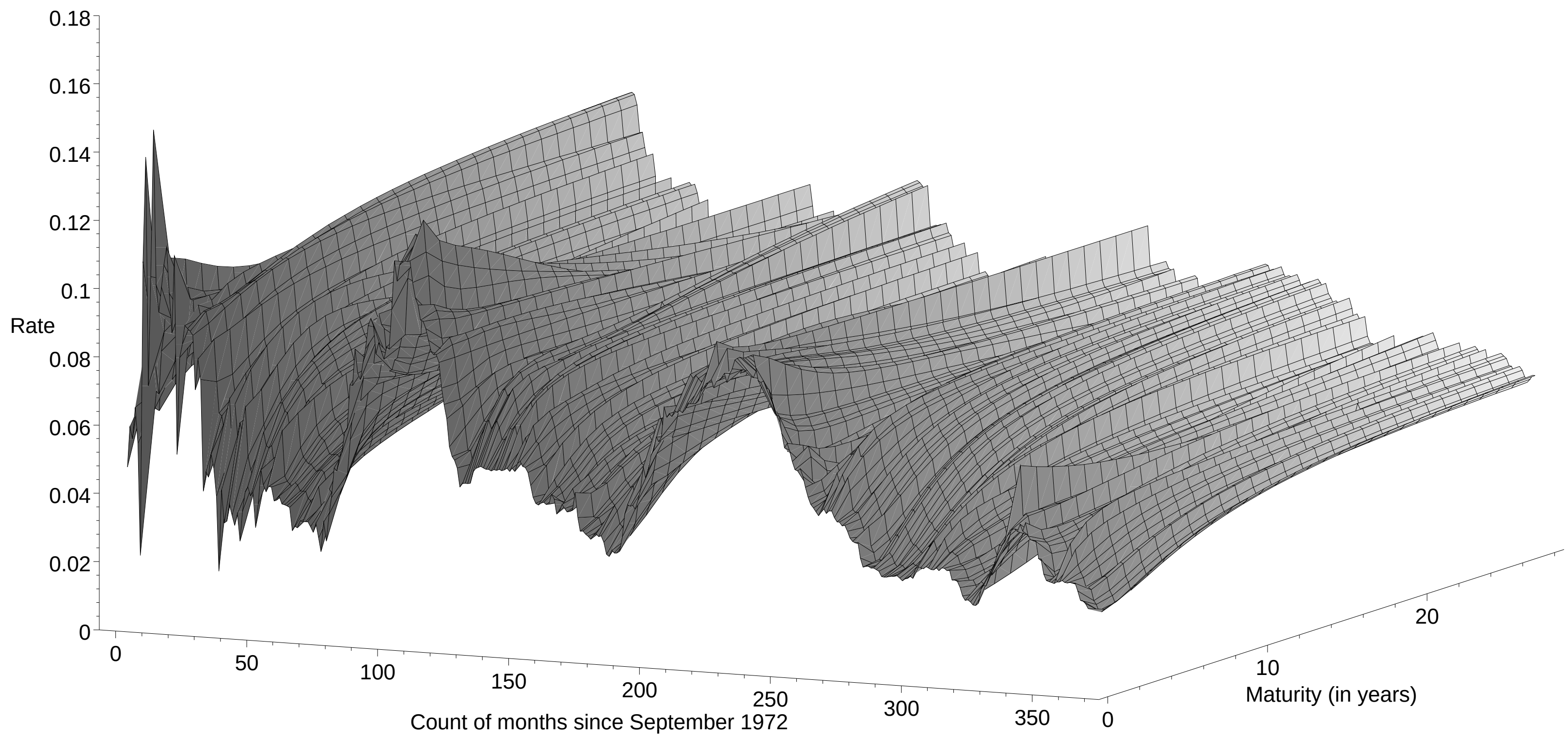


Figure 7: Historical yields of the German debt securities market

5.1.3 The present value of a deterministic cash flow

- Discrete time axis $\mathbb{T} = \{t_1, \ldots, t_n\}$, $t_1 < \ldots < t_n$
- **Deterministic cash flow:** $X_{\mathbb{T}} = (X_{t_1}, \ldots X_{t_n}) \in \mathbb{R}^n$, i.e. at time t_i one has the fixed (deterministic) payoff X_{t_i}.
- Under condition (NA) the present value of the cashflow X at time 0 is
$$PV_0(X_{\mathbb{T}}) = \sum_{k=1}^{n} p(0, t_k) X_{t_k}. \tag{50}$$
- Cf. Equation (5)

5.2 Traditional contracts with stochastic interest rates

- LI-contract for i given by two cash flows $({}^i\gamma_t)_{t\in\mathbb{T}} = (\frac{{}^iC_t}{S_t^0}e_0)_{t\in\mathbb{T}}$ and $({}^i\delta_t)_{t\in\mathbb{T}} = (\frac{{}^iD_t}{S_t^0}e_0)_{t\in\mathbb{T}}$ with $\mathbb{T} = \{0, 1, \ldots, T\}$ in years.
- ${}^i\gamma_t = {}^i\delta_t = 0$ for t greater than some $T_i \in \mathbb{T}$, i.e. contract has an expiration date T_i, and each ${}^iC_t$ for $t \le T_i$ given by ${}^iC_t(f, b) = {}^ic\,{}^i\beta_t^\gamma(b^i)$ for all $(f, b) = (f, b^1, b^2, \ldots) \in M$ where ${}^ic$ a positive constant. Let $({}^i\delta_t)_{t\in\mathbb{T}}$ be defined analogously with the variables ${}^iD_t, {}^id$ and ${}^i\beta_t^\delta$. Suppose that ${}^i\beta_t^{\gamma(\delta)}$ is $\mathcal{B}_t^i$-measurable with ${}^i\beta_t^{\gamma(\delta)}(b^i) \in \{0, 1\}$ for all $b^i \in B^i$ $(t \le T_i)$.
- e_0/S_t^0 can be interpreted as the guaranteed payoff of one currency unit at time t = ZCB with maturity t.

1. Term insurance.

- For $t \leq T_i$ one has ${}^i\beta_t^\gamma = 1$ iff (=if and only if) the i-th individual has died in $(t-1, t]$ and for $t < T_i$ that ${}^i\beta_t^\delta = 1$ iff i is still alive at t, but ${}^i\beta_{T_i}^\delta \equiv 0$. Assume that i is alive at $t = 0$.

- Contract is a term insurance with fixed annual premiums ${}^i d$ and the benefit ${}^i c$ in the case of death.

- Note that ${}_{t-1|1}q_x = \mathbf{E}_{\mathbb{B}}[{}^i\beta_t^\gamma]$ $(t > 0)$ and ${}_t p_x = \mathbf{E}_{\mathbb{B}}[{}^i\beta_t^\delta]$ $(0 < t < T_i)$ for an individual of age x (cf. Section 2.4); for convenience reasons, the notation ${}_{-1|1}q_x = 0$ and ${}_0 p_x = 1$ is used.

- The hedge H^* for ${}^i\delta_t - {}^i\gamma_t$ is for $t < T_i$ given by the number of $({}^i c\, {}_{t-1|1}q_x - {}^i d\, {}_t p_x)$ ZCB with maturity t, and for $t = T_i$ by ${}^i c\, {}_{T_i-1|1}q_x$ ZCB with maturity T_i.

2. Endowment.

- Assume for $t < T_i$ that ${}^i\beta_t^\gamma = 1$ if and only if the i-th individual has died in $(t-1, t]$, but ${}^i\beta_{T_i}^\gamma = 1$ if and only if i has died in $(T_i - 1, T_i]$ or is still alive at T_i. Furthermore, ${}^i\beta_t^\delta = 1$ if and only if the i-th individual is still alive at $t < T_i$, but ${}^i\beta_{T_i}^\delta \equiv 0$. Assume that i is alive at $t = 0$.

- Contract is a **endowment** that features fixed annual premiums ${}^i d$ and the benefit ${}^i c$ in the case of death, but also the payoff ${}^i c$ when i is alive at T_i.

- The hedge H^* due to ${}^i\delta_t - {}^i\gamma_t$ is for $t < T_i$ given by the number of $({}^i c \; {}_{t-1|1}q_x - {}^i d \; {}_t p_x)$ ZCB with maturity t, and for $t = T_i$ by ${}^i c \left({}_{T_i - 1|1}q_x + {}_{T_i}p_x \right)$ ZCB with maturity T_i.

All hedging can be done by zero-coupon bonds (matching).

5.3 Historical pricing and valuation

- Consider the contracts from Subsection 5.2.
- Due to the Equivalence Principle (32), we demand

$$\pi_0 \left(\sum_{t=0}^{T_i} {}^i c \, {}^i\beta_t^{\gamma} e_0 / S_t^0 \right) = \pi_0 \left(\sum_{t=0}^{T_i} {}^i d \, {}^i\beta_t^{\delta} e_0 / S_t^0 \right). \qquad (51)$$

- (25) is applied for premium calculation, hence

$$\frac{{}^i d}{{}^i c} = \sum_{t=0}^{T_i} p(0,t) \cdot \mathbf{E}_{\mathbb{B}}[{}^i\beta_t^{\gamma}] \Big/ \sum_{t=0}^{T_i} p(0,t) \cdot \mathbf{E}_{\mathbb{B}}[{}^i\beta_t^{\delta}]. \qquad (52)$$

- (52) (minimum fair premium/benefit) depends on the ZCB prices (or yield curve) at time 0, i.e. ${}^i d/{}^i c$ varies from day to day.

- There is a yield curve given for any time t of the considered historical time axis.

$\Rightarrow$ It is possible to compute the historical value of ${}^i d/{}^i c$ for t (the date when the respective contract was signed) via (48) and (52).

- One obtains

$$\frac{{}^i d}{{}^i c}(t) = \sum_{\tau=0}^{T_i} p(t,\tau)\,{}_{\tau-1|1}q_x(t) \Big/ \sum_{\tau=0}^{T_i-1} p(t,\tau)\,{}_{\tau}p_x(t) \tag{53}$$

for the term insurance and

$$\frac{{}^i d}{{}^i c}(t) = \left(p(t,T_i)\,{}_{T_i}p_x(t) + \sum_{\tau=0}^{T_i} p(t,\tau)\,{}_{\tau-1|1}q_x(t) \right) \Big/ \sum_{\tau=0}^{T_i-1} p(t,\tau)\,{}_{\tau}p_x(t) \tag{54}$$

for the endowment (cf. Example 5.2).

- Consider a man of age $x = 30$ years and the time axis $\mathbb{T} = \{0, 1, \ldots, 10\}$ (in years).
- In Figure 8, the rescaled quotients (53) and (54) are plotted for the above setup.
- The absolute values at the starting point (September 1972) are ${}^{i}d/{}^{i}c = 0.063792$ for the endowment, respectively ${}^{i}d/{}^{i}c = 0.001587$ for the term insurance.
- The plot shows the dynamics of the quotients and hence of the minimum fair premiums ${}^{i}d$ if the benefit ${}^{i}c$ is assumed to be constant.
- The premiums of the endowment seem to be much more subject to the fluctuations of the interest rates than the premiums of the term insurance.

Figure 8: Rescaled plot of the quotient ${}^{i}d/{}^{i}c$ (minimum fair annual premium/benefit) for the 10-years endowment (solid), resp. term insurance (dashed), for a 30 year old man

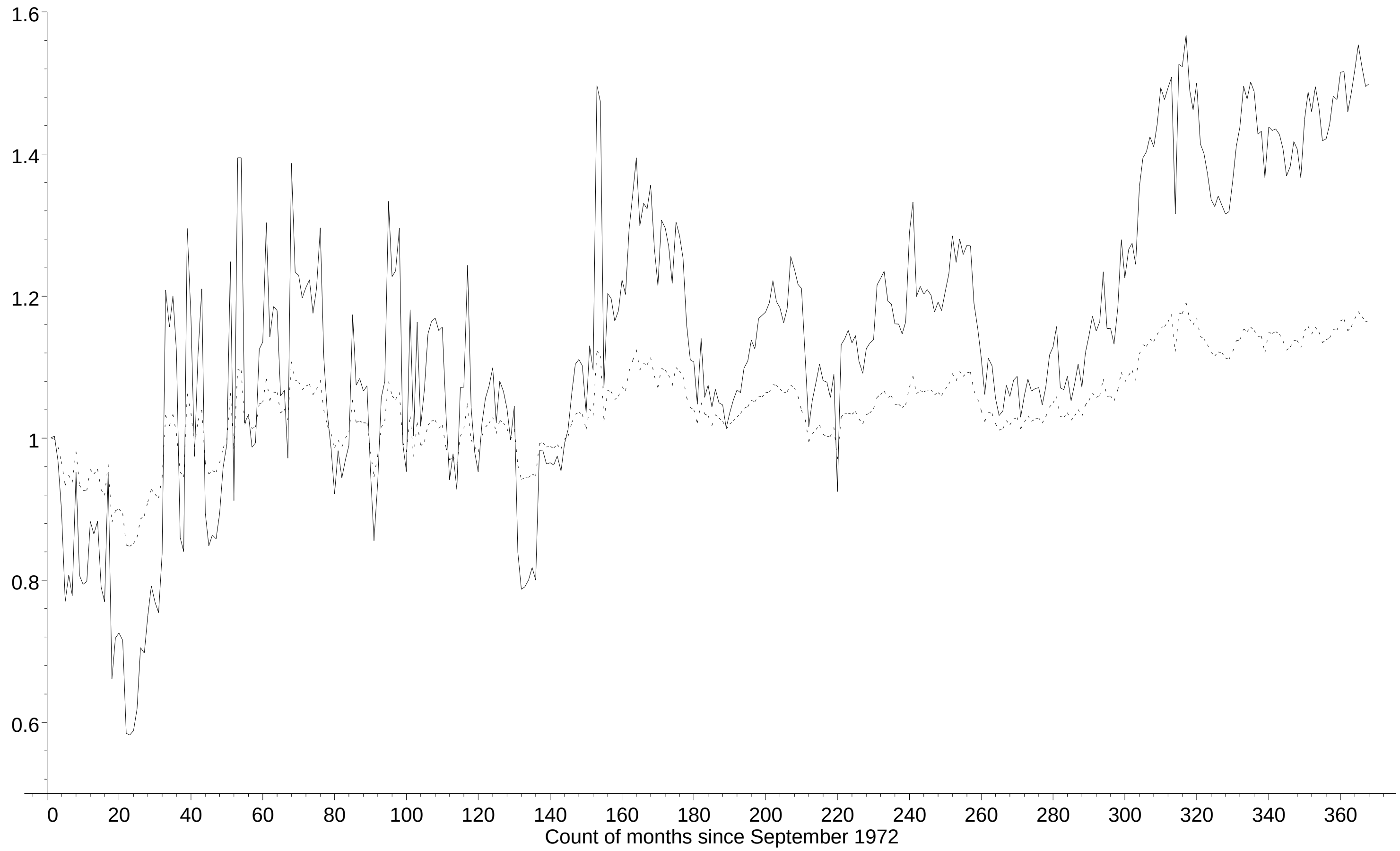


Figure 9: Rescaled plot of the quotient ${}^{i}d/{}^{i}c$ (minimum fair annual premium/benefit) for the 25-years endowment (solid), resp. term insurance (dashed), for a 30 year old man

- Insurance companies do not determine the prices for products daily. Financial risks can emerge as the contracts may be over-valued.

- If one assumes a discrete technical (= first order) rate of interest R'_{tech}, e.g. 0.035, one can compute technical quotients ${}^{i}d_{\text{tech}}/{}^{i}c$ by computing the technical values of zero-coupon bonds, i.e. $p_{\text{tech}}(t,\tau) = (1 + R'_{\text{tech}})^{-\tau}$, and plugging them into (53), resp. (54).

- If a life insurance company charges the technical premiums ${}^{i}d_{\text{tech}}$ instead of the minimum fair premiums ${}^{i}d$ and if one uses the valuation principle (25), the *present value* of the considered insurance contract at time t is

$$ {}^{i}PV = ({}^{i}d_{\text{tech}} - {}^{i}d) \cdot \sum_{\tau=0}^{T_i - 1} p(t,\tau)\, {}_{\tau}p_x(t) \tag{55} $$

  due to the Principle of Equivalence, respectively (51).

- In particular, this means that the insurance company can book the gain or loss (55) in the mean (or limit) at time 0 as long as proper risk management (as described in Section 5.2) takes place afterwards.

- The present value (55) is a measure for the profit, or simply *the expected discounted profit* of the considered contract if one neglects all additional costs and the fact that in this specific example first order mortality tables are used.

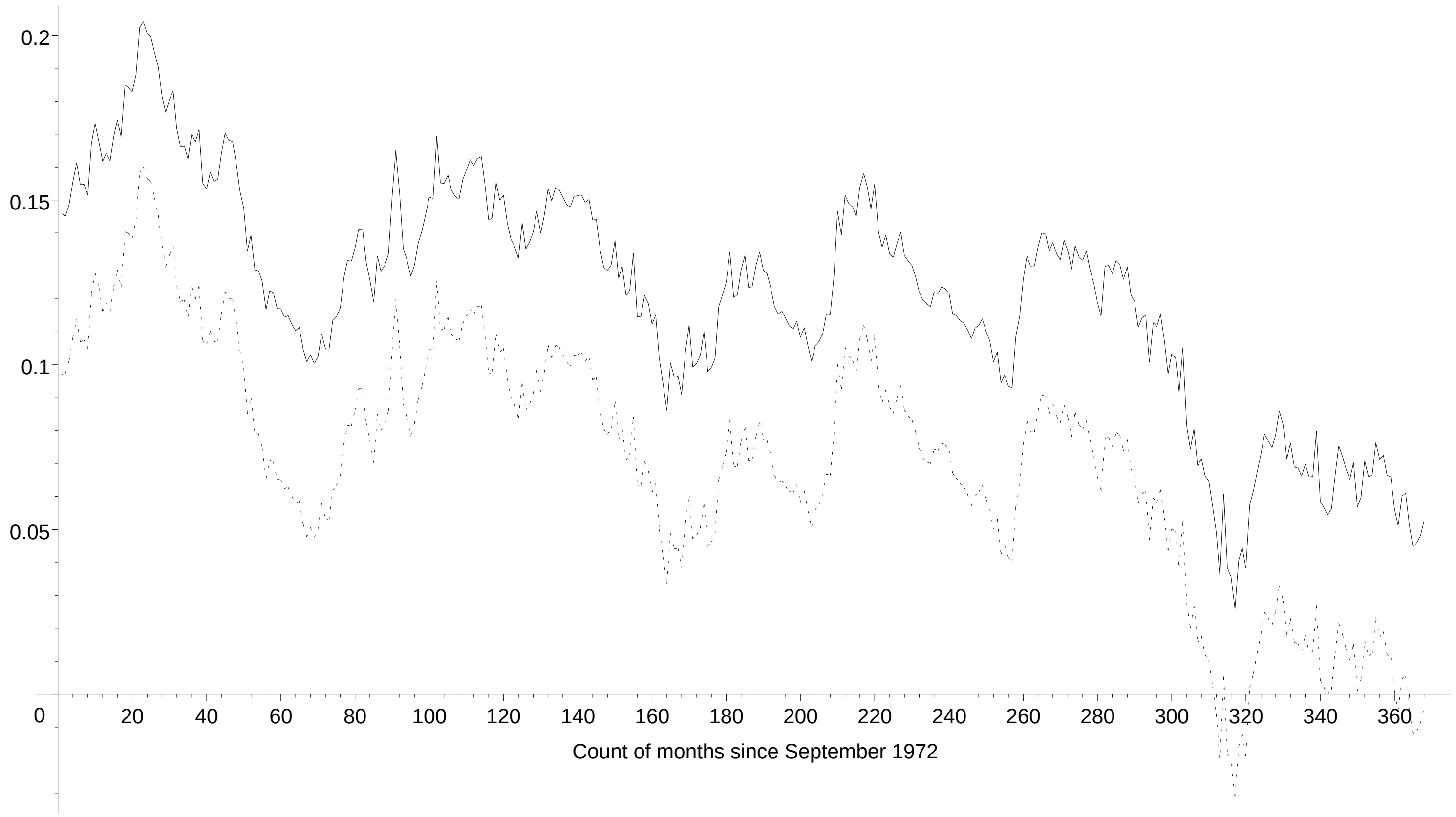


Figure 10: ${}^{i}PV/{}^{i}c$ (present value/benefit) for the 10-years endowment under a technical interest rate of 0.035 (solid) and 0.050 (dashed) for a 30 year old man

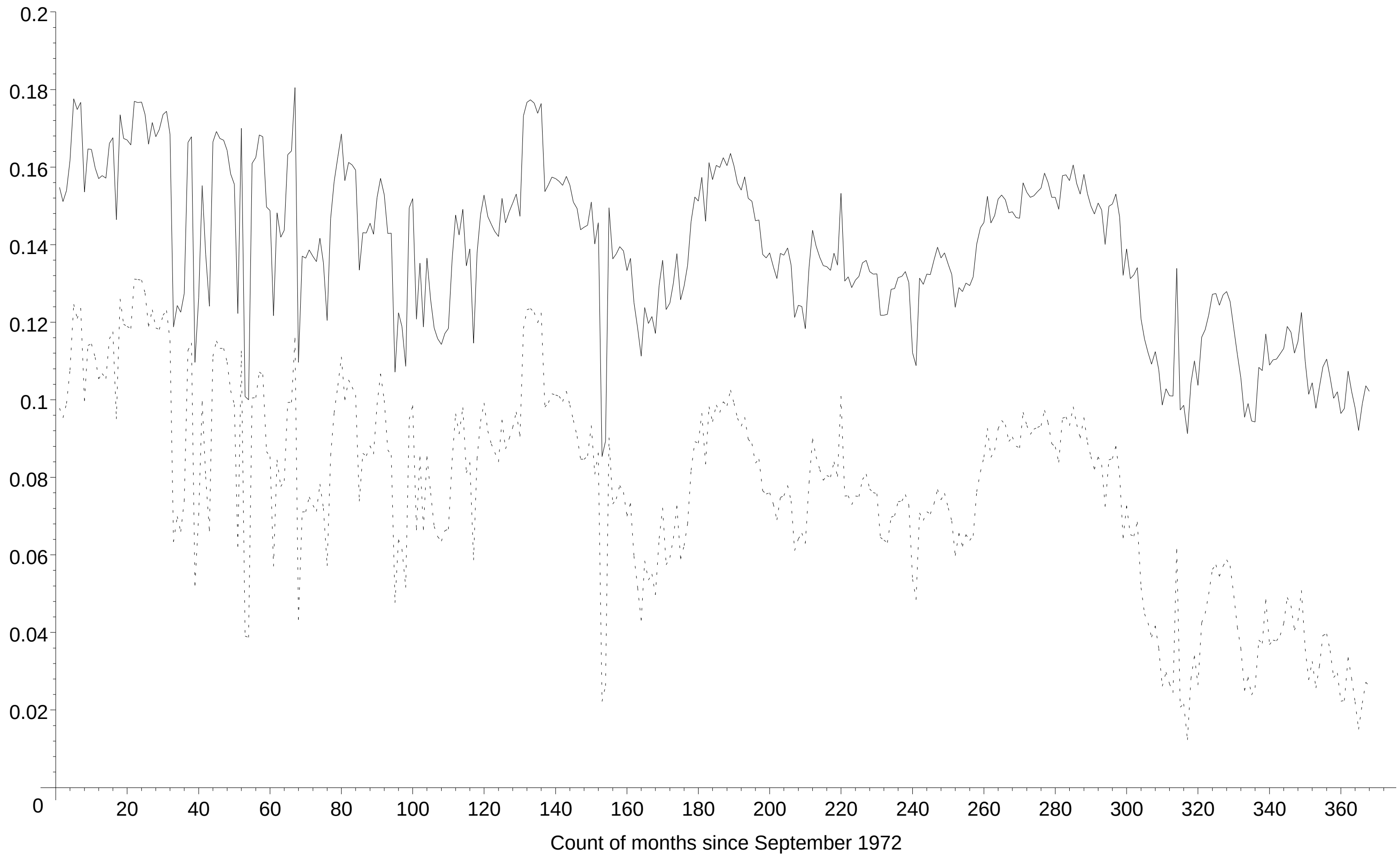


Figure 11: ${}^{i}PV/{}^{i}c$ (present value/benefit) for the 25-years endowment under a technical interest rate of 0.035 (solid) and 0.050 (dashed) for a 30 year old man

Next two tables:

Selected (extreme) values due to different contracts for a 30 year old man (fixed benefit: ${}^{i}c = 100,000$ Euros)

Date	1974/07/31	1999/01/31
Term insurance: 10 years		
Techn. premium ${}^{i}d_{\text{tech}}$ ($R'_{\text{tech}} = 0.035$)	168.94	
Techn. premium ${}^{i}d_{\text{tech}}$ ($R'_{\text{tech}} = 0.050$)	165.45	
Minimum fair annual premium ${}^{i}d$	152.46	168.11
Present value ${}^{i}V$ ($R'_{\text{tech}} = 0.035$)	108.90	7.17
Present value ${}^{i}V$ ($R'_{\text{tech}} = 0.050$)	85.84	-22.80
Endowment: 10 years		
Techn. premium ${}^{i}d_{\text{tech}}$ ($R'_{\text{tech}} = 0.035$)	8,372.65	
Techn. premium ${}^{i}d_{\text{tech}}$ ($R'_{\text{tech}} = 0.050$)	7,706.24	
Minimum fair annual premium ${}^{i}d$	5,285.55	8,072.26
Present value ${}^{i}V$ ($R'_{\text{tech}} = 0.035$)	20,398.70	2,578.55
Present value ${}^{i}V$ ($R'_{\text{tech}} = 0.050$)	15,995.27	-3,141.95

Date	1974/07/31	1999/01/31
Term insurance: 25 years		
Techn. premium ${}^{i}d_{\text{tech}}$ ($R'_{\text{tech}} = 0.035$)	328.02	
Techn. premium ${}^{i}d_{\text{tech}}$ ($R'_{\text{tech}} = 0.050$)	303.27	
Minimum fair annual premium ${}^{i}d$	216.37	303.90
Present value ${}^{i}V$ ($R'_{\text{tech}} = 0.035$)	1,009.56	376.84
Present value ${}^{i}V$ ($R'_{\text{tech}} = 0.050$)	785.80	-9.83
Endowment: 25 years		
Techn. premium ${}^{i}d_{\text{tech}}$ ($R'_{\text{tech}} = 0.035$)	2,760.85	
Techn. premium ${}^{i}d_{\text{tech}}$ ($R'_{\text{tech}} = 0.050$)	2,255.93	
Minimum fair annual premium ${}^{i}d$	808.39	2,177.32
Present value ${}^{i}V$ ($R'_{\text{tech}} = 0.035$)	17,655.42	9,118.39
Present value ${}^{i}V$ ($R'_{\text{tech}} = 0.050$)	13,089.53	1,228.34

5.4 Unit-linked pure endowment with guarantee

- LI-contract for i given by $({}^i\gamma_t)_{t\in\mathbb{T}} = ({}^iC_t e_1)_{t\in\mathbb{T}}$ and $({}^i\delta_t)_{t\in\mathbb{T}} = (\frac{{}^iD_t}{S_t^0} e_0)_{t\in\mathbb{T}}$ with $\mathbb{T} = \{0, 1, \ldots, T\}$ in years.
- Assume for $t < T_i$ that ${}^i\beta_t^\gamma = 0$ and ${}^i\beta_{T_i}^\gamma = 1$ if and only if i is still alive at T_i. Furthermore, ${}^i\beta_t^\delta = 1$ if and only if the i-th individual is still alive at $t < T_i$, but ${}^i\beta_{T_i}^\delta \equiv 0$.
- Let ${}^iD_t$ be as in Section 5.2 on page 88.
- Let E be the non-random number of shares of type 1 and $G > 0$ be the guaranteed minimum payoff which are paid if i is alive at T_i.

$\Rightarrow$ ${}^ic_{T_i} = \max\{G/S_{T_i}^1, E\}$ and ${}^iC_{T_i} = {}^ic_{T_i}\,{}^i\beta_{T_i}^\gamma$

- The contract is a **unit-linked pure endowment with guarantee** that features fixed annual premiums ${}^i d$ and the benefit ${}^i c_{T_i} S^1_{T_i}$ when i is alive at T_i.

- The hedge H^* due to ${}^i\delta_t - {}^i\gamma_t$ is for $t < T_i$ given by the number of $-{}^i d \, {}_t p_x$ ZCB with maturity t, and for $t = T_i$ by ${}_{T_i}p_x G$ ZCB and ${}_{T_i}p_x E$ European Calls with underlying S^1, strike price $K = G/E$ and maturity T_i.

5.5 Premium and reserve with CRR (spreadsheet)

`www.mathematik.tu-darmstadt.de/~tfischer/`
`Unit-linkedPureEndowment+Guarantee.xls`

- For the numeric example we use the Cox-Ross-Rubinstein model as in Section 3.8, but here with $\mathbb{T} = \{0, 1, \ldots, 10\}$ in years.
- Understand the computation of the minimum fair premium.
- With $E = 1000$ and $G = 140000.00$ we obtain ${}^{i}d = 12257.38$ as minimum fair premium.
- What is a reasonable definition for the reserve in the modern framework? (Cf. Equation (16))
- Try to understand the computation of the reserve R^a. What is the premium part of the reserve?
- Explain why recursion formula (18) cannot be used here.

6 Conclusion

- Reasonable brief system of axioms for modern life insurance exists
- Adaption of classical convergence-idea (SLLN) possible
- Minimum fair price uniquely determined by axioms
- Modern valuation and hedging crucial for real companies
- Classical life insurance mathematics a special case of the modern approach

7 Appendix

7.1 Stochastic independence and product spaces

- Independence of random variables X, Y means that they don't influence each other.
- Precise: If X, Y are real-valued on $(B, \mathcal{B}, \mathbb{B})$, then X and Y are **stochastically independent** if and only if for each pair of Borel sets $A_1, A_2 \subset \mathbb{R}$

$$\mathbb{B}(X \in A_1 \text{ and } Y \in A_2) = \mathbb{B}(X \in A_1) \cdot \mathbb{B}(X \in A_1). \tag{56}$$

- **'Real-world' example:** Two coins, X and Y can have the states 0 (the one side) or 1 (the other side) with probability $\frac{1}{2}$. Clearly, $\mathbb{B}(X = 0, Y = 0) = \frac{1}{4}$.
- Construction of independent random variables by product spaces.

- Given two probability spaces $(B^1, \mathcal{B}^1, \mathbb{B}^1)$ and $(B^2, \mathcal{B}^2, \mathbb{B}^2)$, there exists a (uniquely determined) probability space

$$(B^1 \times B^2, \mathcal{B}^1 \otimes \mathcal{B}^2, \mathbb{B}^1 \otimes \mathbb{B}^2) \tag{57}$$

which distributes to all events of form $A_1 \times A_2$ with $A_1 \in \mathcal{B}^1$ and $A_2 \in \mathcal{B}^2$ the probability

$$\mathbb{B}^1 \otimes \mathbb{B}^2(A_1 \times A_2) = \mathbb{B}^1(A_1) \cdot \mathbb{B}^2(A_2). \tag{58}$$

- When X is defined on $(B^1, \mathcal{B}^1, \mathbb{B}^1)$ and Y on $(B^2, \mathcal{B}^2, \mathbb{B}^2)$, then these random variables are independent on the common space (57) (where $X(b_1, b_2) := X(b_1)$ and $Y(b_1, b_2) := Y(b_2)$).

- **'Real-world' example:** The two coins! Here, $B^i = \{0, 1\}$, $B^1 \times B^2 = \{(0,0), (1,0), (0,1), (1,1)\}$, X and Y defined as Id. $A_1 = \{0\}$, $A_2 = \{0\}$ reflects the independence example above.

- A generalization to infinite products is possible.

7.2 A corollary of Fubini's Theorem

COROLLARY 7.1. *Consider two probability spaces* $(F, \mathcal{F}, \mathbb{F})$ *and* $(B, \mathcal{B}, \mathbb{B})$ *and a* $\mathbb{F} \otimes \mathbb{B}$*-integrable real-valued random variable* X *on* $F \times B$*. Then*

$$\mathbf{E}_{\mathbb{F}\otimes\mathbb{B}}[X] = \mathbf{E}_{\mathbb{F}}[\mathbf{E}_{\mathbb{B}}[X]] = \mathbf{E}_{\mathbb{B}}[\mathbf{E}_{\mathbb{F}}[X]]. \tag{59}$$

- The order of integration can be chosen arbitrarily.
- In particular, $\mathbf{E}_{\mathbb{F}}[X]$ (resp. $\mathbf{E}_{\mathbb{B}}[X]$) exists $\mathbb{B}$-a.s. (resp. $\mathbb{F}$-a.s.).

7.3 The Strong Law of Large Numbers

- Recall that on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ **almost surely (a.s.)** means **on a set with measure/probability 1**.
- A sequence of real valued random variables $(X_n)_{n\in\mathbb{N}}$ is said to fulfill the **Strong Law of Large Numbers** whenever

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - \mathbf{E}[X_i]) \overset{n\to\infty}{\longrightarrow} 0 \quad a.s. \tag{60}$$

- Two important results on the Strong Law of Large Numbers by Kolmogorov

THEOREM 7.2 (Kolmogorov/Etemadi). *Any sequence of real valued, integrable, identically distributed and pairwise independent random variables* $(X_n)_{n\in\mathbb{N}}$ *fulfills the Strong Law of Large Numbers.*

"Real-world example": Fair gambling dice with numbers from 1 to 6. The arithmetic mean of the results will always converge to 3.5.

THEOREM 7.3 (Kolmogorov's Criterion). *Any sequence of real valued, integrable and independent random variables* $(X_n)_{n\in\mathbb{N}}$ *with*

$$\sum_{i=1}^{\infty} \frac{1}{n^2} Var(X_i) < \infty \tag{61}$$

fulfills the Strong Law of Large Numbers.

"Real-world example": Life insurance! (see Section 1.3.1)

7.4 Conditional expectations and martingales

- Let Y be a $\mathcal{B}_T$-measurable integrable random variable on the filtered probability space $(B, (\mathcal{B}_t)_{t\in\mathbb{T}}, \mathbb{B})$
- $Z = \mathbf{E}[Y|\mathcal{B}_t]$, the **conditional expectation** of Y given $\mathcal{B}_t$, is the a.s.-uniquely determined $\mathcal{B}_t$-measurable random variable, such that

$$\int_C Z d\mathbb{B} = \int_C Y d\mathbb{B} \quad \forall C \in \mathcal{B}_t \tag{62}$$

  i.e. Z is the "smoothing" of Y with respect to $\mathcal{B}_t$
- Special cases
  1. $\mathbf{E}[Y|\mathcal{B}_0] = \mathbf{E}[Y]$ a.s.
  2. $\mathbf{E}[Y|\mathcal{B}_T] = Y$ a.s.

- When C is a minimal element of $\mathcal{B}_t$, i.e. when it contains no other element of $\mathcal{B}_t$, and $\mathbb{B}(C) > 0$, then

$$\mathbf{E}[Y|\mathcal{B}_t](b) = \mathbf{E}[Y|C] := \frac{1}{\mathbb{B}(C)} \int_C Y d\mathbb{B} \tag{63}$$

for any $b \in C$.

- Some rules of calculus
  1. $\mathbf{E}[\mathbf{E}[Y|\mathcal{B}_t]|\mathcal{B}_s] = \mathbf{E}[Y|\mathcal{B}_s]$ a.s. for $s < t$
  2. $\mathbf{E}[XY|\mathcal{B}_t]| = X \cdot \mathbf{E}[Y|\mathcal{B}_t]$ a.s. if X $\mathcal{B}_t$-measurable and XY integrable
  3. $\mathbf{E}[aX + bY|\mathcal{B}_t] = a\mathbf{E}[X|\mathcal{B}_t] + b\mathbf{E}[Y|\mathcal{B}_t]$

EXAMPLE 7.4 (Conditional expectations).

- You have to pass two exams ($t = 1, 2$) to get the maths diploma.
- Your auntie gives you 1 EUR when you get the diploma (r.v. X).
- The expected value of the gift (i.e. of X) conditioned on the information given at time t is $\mathbf{E}[X|\mathcal{B}_t]$. Here,
  $B = \{pp, pf, ff\}$
  $\mathcal{B}_0 = \{\emptyset, B\}, \quad \mathcal{B}_1 = \{\emptyset, \{pp, pf\}, \{ff\}, B\}, \quad \mathcal{B}_2 = \mathcal{P}(B) = \{\emptyset, \{pp\}, \{pf\}, \{ff\}, \{pp, pf\}, \{pf, ff\}, \{pp, ff\}, B\}$

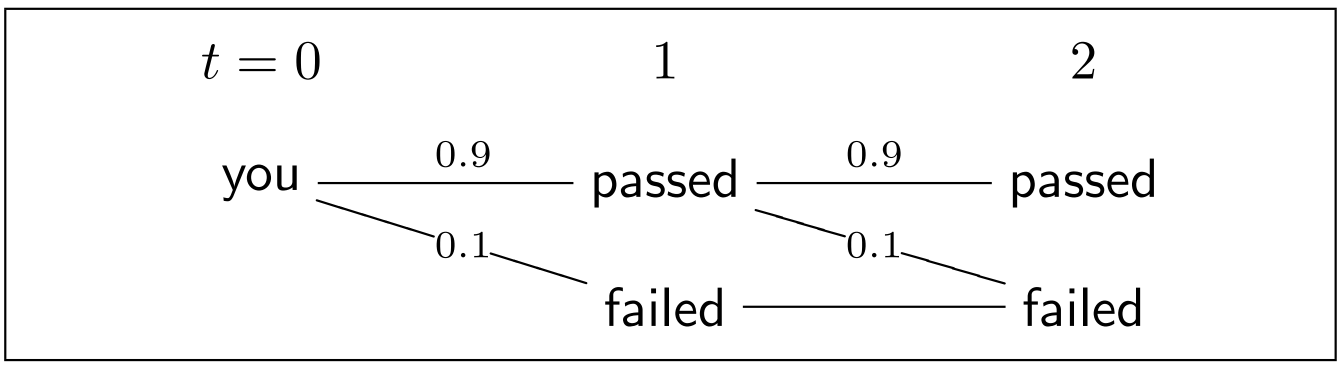


Figure 12: The situation with probabilities

Example 7.4 (continued)

- $\mathbf{E}[X|\mathcal{B}_0] = \mathbf{E}[X] = 0.81$
- $\mathbf{E}[X|\mathcal{B}_1](\{pp, pf\}) = \mathbf{E}[X|\mathcal{B}_1](p*) = 0.9$ and $\mathbf{E}[X|\mathcal{B}_1](f*) = 0$
- $\mathbf{E}[X|\mathcal{B}_2](pp) = 1$ and $\mathbf{E}[X|\mathcal{B}_2](\{pf, ff\}) = \mathbf{E}[X|\mathcal{B}_1](*f) = 0$, i.e. $\mathbf{E}[X|\mathcal{B}_2] = X$
- Observe: $\mathbf{E}[X|\mathcal{B}_1]$(passed/failed at 1) is exactly the expectation of X you would compute at 1 and in this state. This is the meaning of (63).

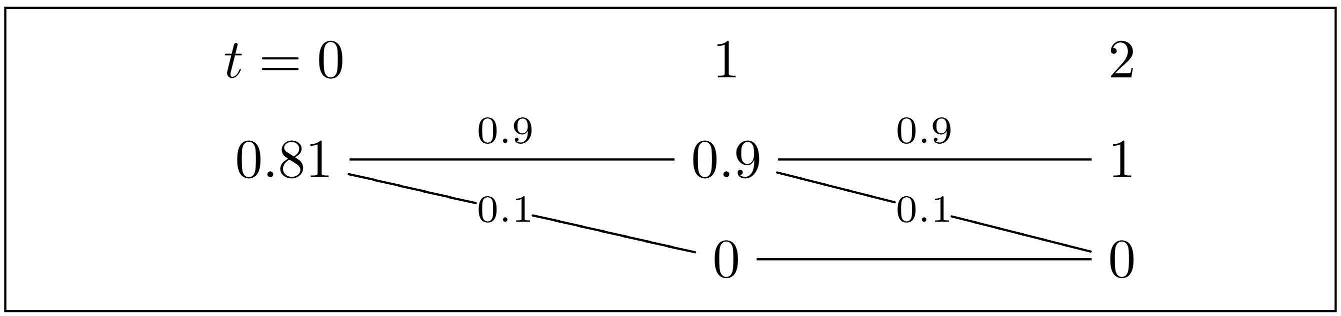


Figure 13: (Expected) Value process of your aunties gift

- An **adapted stochastic process** (with respect to the filtration $(\mathcal{B}_t)_{t\in\mathbb{T}}$) is a vector $X = (X_t)_{t\in\mathbb{T}}$ of $\mathcal{B}_t$-measurable random variables X_t.
- A **martingale** is an adapted stochastic process $X = (X_t)_{t\in\mathbb{T}}$ such that for $s \leq t$

$$\mathbf{E}[X_t|\mathcal{B}_s] = X_s \text{ a.s.} \quad (s, t \in \mathbb{T}). \tag{64}$$

- **Example:** $(\mathbf{E}[Y|\mathcal{B}_t])_{t\in\mathbb{T}}$ is a martingale (Y $\mathcal{B}_T$-measurable).

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