Introduction to Hybridized Discontinuous Galerkin (HDG) Methods

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(Not so) Brief Overview

Steady state problem

General second order PDE

$u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ ($\Omega \subset \mathbb{R}^d$, $d \geq 1$ bounded open domain)

$$\begin{aligned} -\nabla \cdot \nu \nabla u + \nabla \cdot (\mathbf{b} u) &= f \qquad \text{in } \Omega\,, \\ u &= 0 \qquad \text{on } \Gamma\,. \end{aligned}$$

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Possible discretizations

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1. Finite Difference Method: use $\partial_x u(x_i) \approx \frac{u(x_i+h)-u(x_i)}{h}$ with "fixed" h

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2. Finite Volume Method: use the integral form of the equation
3. Finite Element Method: use the weak form

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- not conservative
- poor performance for convection dominated problems

Weak form & Classical Galerkin

Weak form

$$-\nabla \cdot \nu \nabla u + \nabla \cdot (\mathbf{b} u) = f \qquad \text{in } \Omega$$

multiply by v and IBP

$$\underbrace{\int_\Omega \nu \nabla u \cdot \nabla v - \int_\Omega u\mathbf{b} \cdot \nabla v + BC}_{a(u,v)} = \underbrace{\int_\Omega fv}_{l(v)}$$

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$$u, v \in V = H_0^1(\Omega) = \{v \in L^2(\Omega) : \nabla v \in [L^2(\Omega)]^d,\ v|_\Gamma = 0\}$$

Finite Element Methods

$$?u \in V : a(u, v) = l(v) \quad \forall v \in V$$

- V is infinite dimensional

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- Easy to go for high polynomial degree

Linear system

Choose a basis of $V_h : \{\Phi_i, \cdots, \Phi_N\}$

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- $\mathbf{A}_{i,j} = a(\Phi_j, \Phi_i)$

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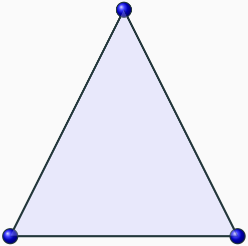
- Direct or iterative solver?
- Size vs condition number

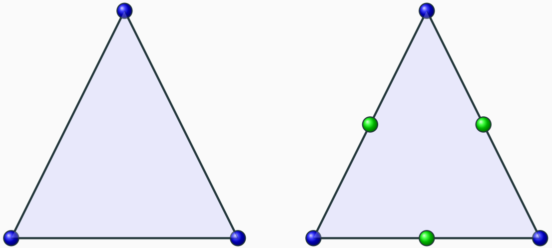
Degrees of Freedoms in 2D

$k = 1$



Degrees of Freedoms in 2D

$k = 1$ $\qquad$ $k = 2$



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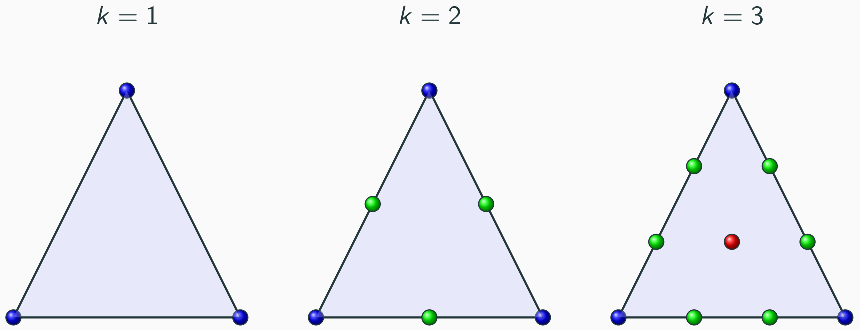

Diffusion Dominated Problem

Consider the advection-diffusion problem

$$\begin{aligned} -\kappa \Delta u + \vec{c} \cdot \nabla u &= f \qquad \text{in } \Omega = [0,1] \times [0,1], \\ u &= g_D \qquad \text{on } \Gamma = \partial\Omega, \end{aligned}$$

with exact solution $u(x,y) = \sin(6x)\sin(6y)$, f and g_D are derived from this exact solution, $\vec{c} = (-1,1)^T$ and κ is the diffusion coefficient.

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Convergence rates

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch^{k+1}$$

$\kappa = 1$

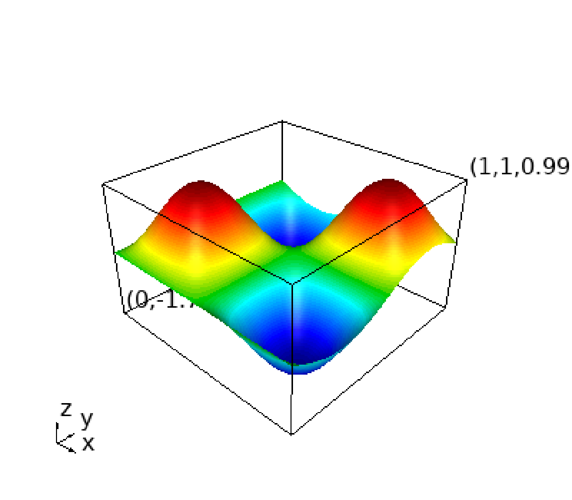

Figure 1: $||u_{CG} - u_e||_{L_2} = 4.7704e - 4$

Pure Advection Problem

Consider the same problem

$$-\kappa \Delta u + \vec{c} \cdot \nabla u = f \qquad \text{in } \Omega = [0,1] \times [0,1],$$
$$u = g_D \qquad \text{on } \Gamma = \partial\Omega,$$

with exact solution $u(x,y) = \sin(6x)\sin(6y)$, f and g_D are derived from this exact solution, $\vec{c} = (-1,1)^T$ and $\kappa = 0$.

$\kappa = 0$

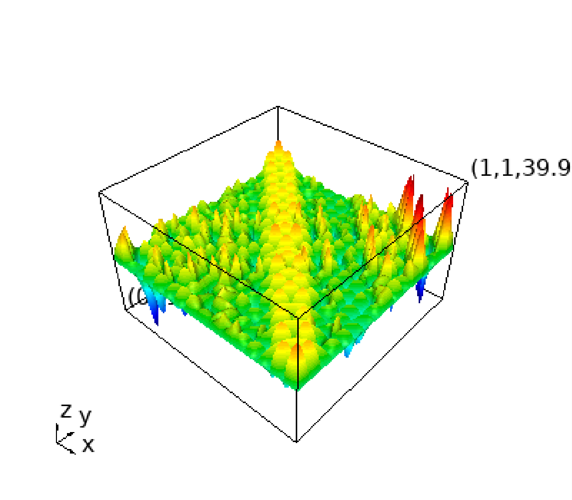

Figure 2: $||u_{CG} - u_e||_{L_2} = 7.02071$

DG

Possible improvement: DG

FVM

- numerical fluxes over the elements
- upwind flux

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CG

- higher order discretization

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DG

- derive weak form starting from one element
- connection between elements via fluxes
- higher order discretization

Mesh first

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Rewrite $-\nabla \cdot \nu \nabla u + \nabla \cdot (\mathbf{b} u) = f$ using $\mathbf{q} = -\nabla u$

First order system

$$\begin{aligned} \nu \nabla \cdot \mathbf{q} + \nabla \cdot (\mathbf{b} u) &= f \\ \mathbf{q} + \nabla u &= \mathbf{0} \end{aligned}$$

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IBP on mesh element K + discretization

$$\begin{aligned} \int_K f w_h &= -\int_K \nu \mathbf{q}_h \cdot \nabla w_h + \int_{\partial K} \nu \widehat{\mathbf{q}}_h \cdot \mathbf{n} w_h \\ &\quad - \int_K u_h \mathbf{b} \cdot \nabla w_h + \int_{\partial K} w_h \widehat{u}_h \mathbf{b} \cdot \mathbf{n} \\ \int_K \mathbf{q}_h \cdot \mathbf{v}_h &= -\int_K \nabla u_h \cdot \mathbf{v}_h + \int_{\partial K} (u_h - \widehat{u}_h) \mathbf{v}_h \cdot \mathbf{n} \end{aligned}$$

1 equation or 2 equations?

If $\mathbf{v} = \nu \nabla w_h$

$$\begin{aligned}\int_K f w_h = \int_K \nu \nabla u_h \cdot \nabla w_h + \int_{\partial K} (\widehat{u}_h - u_h) \nu \nabla w_h \cdot \mathbf{n} + \int_{\partial K} \nu \widehat{\mathbf{q}}_h \cdot \mathbf{n} w_h \\ - \int_K u_h \mathbf{b} \cdot \nabla w_h + \int_{\partial K} w_h \widehat{u}_h \mathbf{b} \cdot \mathbf{n}\end{aligned}$$

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$$- \int_K u_h \mathbf{b} \cdot \nabla w_h + \int_{\partial K} w_h \widehat{u}_h \mathbf{b} \cdot \mathbf{n}$$

Summing over all $K \in \mathcal{T}_h$

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Summing over all $K \in \mathcal{T}_h$

The interior faces will show up twice

Choice of the numerical flux

Advection

Use upwinding

$$\widehat{u}_h = \begin{cases} u_L & \text{if } \mathbf{b} \cdot \mathbf{n} \geq 0 \\ u_R & \text{if } \mathbf{b} \cdot \mathbf{n} < 0 \end{cases}$$

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Plenty of possibilities (see Brezzi-Marini survey)

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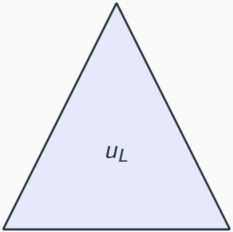
Example with 2 equations: Local DG: $\widehat{\mathbf{q}}_h = \mathbf{q}_L + \tau(u_L - u_R)\mathbf{n}$

Upwind in 2D



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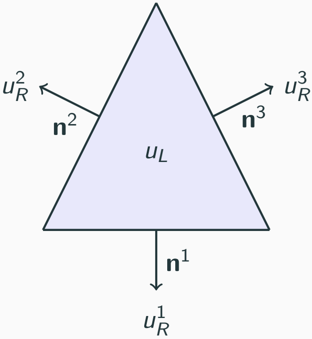

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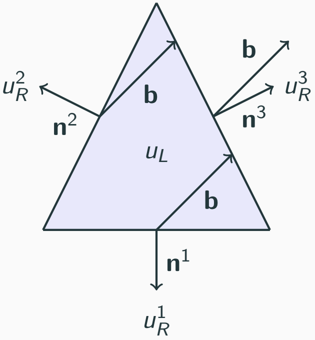

u_R^2
$\mathbf{n}^2$
$\mathbf{b}$
u_L
$\mathbf{b}$
$\mathbf{b}$
u_R^3
$\mathbf{n}^3$
$\mathbf{n}^1$
u_R^1

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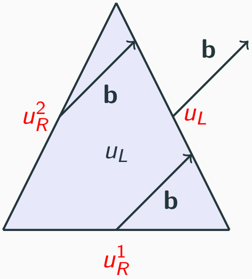

IP discretization

IP approximate weak form

Seek $u_h \in V_h$ such that $a_{DG}(u_h, v_h) = l_{DG}(v_h)$ for all $v_h \in V_h$.

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IP notations

$$a_{DG}(u_h, v_h) = \sum_{K \in \mathcal{T}_h} \int_K \nu \nabla u_h \cdot \nabla w_h - \sum_{K \in \mathcal{T}_h} \int_{\partial K} u_h \mathbf{b} \cdot \nabla v_h$$

$$+ \text{interior face terms}$$

$$l_{DG}(v_h) = \sum_{K \in \mathcal{T}_h} \int_K f v_h + \text{BC}$$

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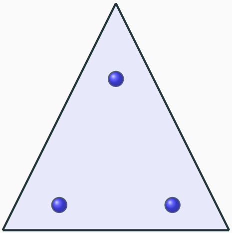
2D DG basis functions

$k = 1$



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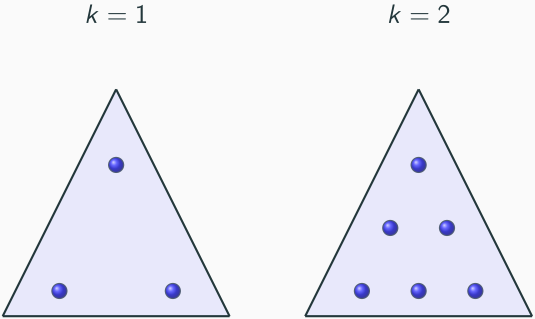

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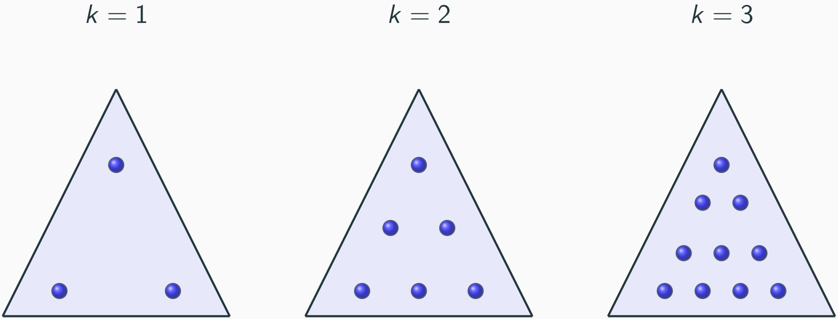

CG vs DG pros and cons

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- lower number of degrees of freedom

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- fails for convection dominated cases
- hard to do *hp* adaptivity: the unknowns on different elements are connected

DG

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- harder to solve with an iterative solver
- works better for convection dominated cases
- *hp* adaptivity is easy: the unknowns on different elements are not connected

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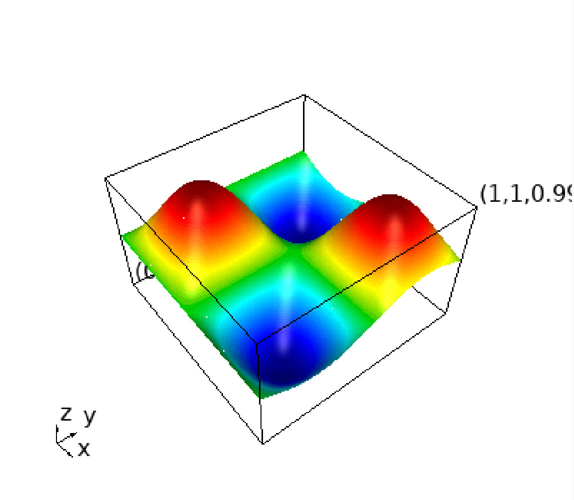

Figure 3: $||u_{DG} - u_e||_{L_2} = 3.8546e - 4$

Pure Advection Problem

Consider the same problem

$$-\kappa\Delta u + \vec{c}\cdot\nabla u = f \qquad \text{in } \Omega = [0,1]\times[0,1],$$
$$u = g_D \qquad \text{on } \Gamma = \partial\Omega,$$

with exact solution $u(x,y) = \sin(6x)\sin(6y)$, f and g_D are derived from this exact solution, $\vec{c} = (-1,1)^T$ and $\kappa = 0$.

$\kappa = 0$

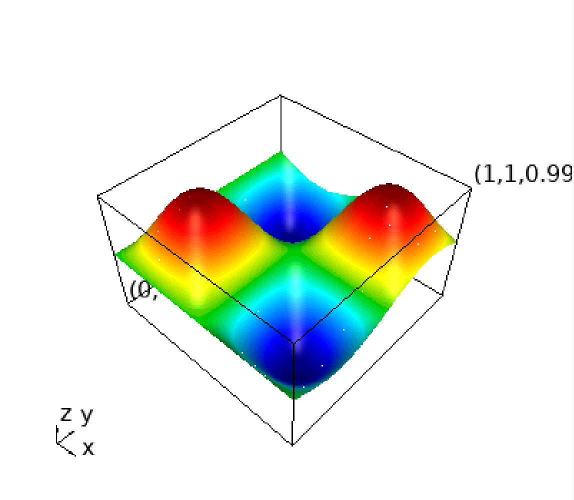

Figure 4: $||u_{DG} - u_e||_{L_2} = 3.0956e - 4$

Idea of HDG

Hybridizable DG

DG

- derive weak form on one element

Hybridizable DG

DG

- derive weak form on one element
- connection between elements via fluxes

Hybridizable DG

DG

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HDG

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Hybridizable DG

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HDG

- derive weak form on one element
- additional unknowns on the edges

Hybridizable DG

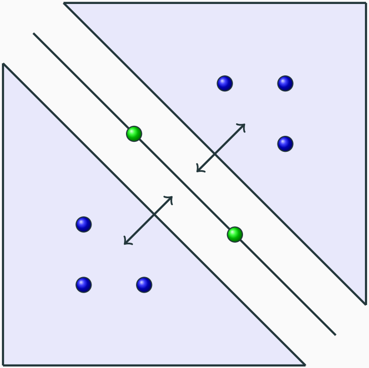
DG

- derive weak form on one element
- connection between elements via fluxes

HDG

- derive weak form on one element
- additional unknowns on the edges
- connection between elements via fluxes that uses functions on the edges

HDG



$$u_h \in V_h = \left\{ v_h \in L^2(\Omega),\ v_h \in P_k(K)\ \forall K \in \mathcal{T}_h \right\}$$

$$\overline{u}_h \in \overline{V}_h = \left\{ \overline{v}_h \in L^2(\mathcal{F}),\ \overline{v}_h \in P_k(F)\ \forall F \in \mathcal{F} \right\}$$

HDG fluxes

DG fluxes

Advection: upwinding

HDG fluxes

DG fluxes

Advection: upwinding

Diffusion Interior penalty or Local DG or one of the many

HDG fluxes

DG fluxes

Advection: upwinding

Diffusion Interior penalty or Local DG or one of the many

HDG fluxes

Advection: $\widehat{u}_h = \begin{cases} u_L & \text{if } \mathbf{b} \cdot \mathbf{n} \geq 0 \\ \overline{u} & \text{if } \mathbf{b} \cdot \mathbf{n} < 0 \end{cases}$

HDG fluxes

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Diffusion IP: $\widehat{\mathbf{q}}_h = \nabla u_L + \dfrac{\alpha}{h}(u_L - \overline{u})\mathbf{n}$

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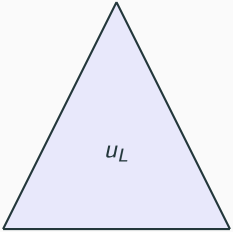
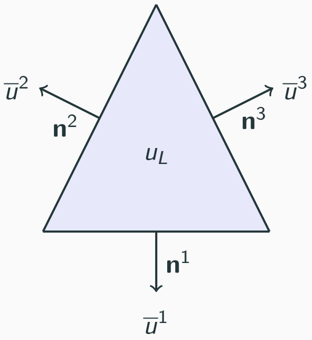
Diffusion IP: $\widehat{\mathbf{q}}_h = \nabla u_L + \dfrac{\alpha}{h}(u_L - \overline{u})\mathbf{n}$

Local DG: $\widehat{\mathbf{q}}_h = \mathbf{q}_L + \tau(u_L - \overline{u})\mathbf{n}$

Upwind for HDG



Upwind for HDG



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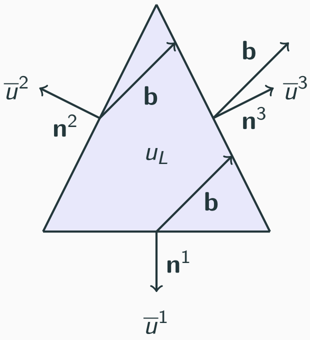
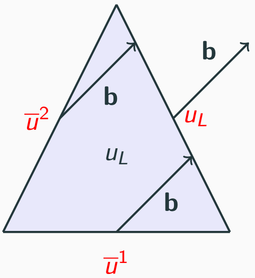


Upwind for HDG



2 unknowns but only 1 equation

Solve $-u'' = 1, u(-1) = u(1) = 0$ as

How to choose $\overline{u}$?

$$
\begin{aligned}
-u'' &= 1 \quad \text{on } (-1, 0) & \qquad -u'' &= 1 \quad \text{on } (0, 1) \\
u(-1) &= 0 & u(0) &= \overline{u} \\
u(0) &= \overline{u} & u(1) &= 0
\end{aligned}
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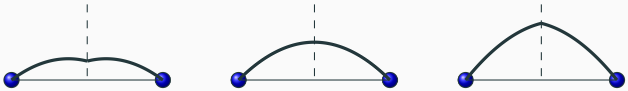
\end{aligned}
$$

$\overline{u} = 0.25$ $\overline{u} = 0.5$ $\overline{u} = 0.75$

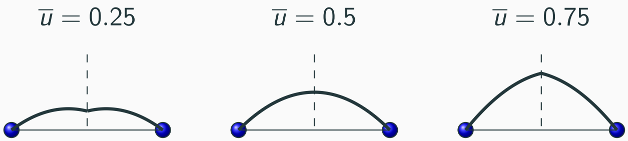


2 unknowns but only 1 equation

Solve $-u'' = 1, u(-1) = u(1) = 0$ as

How to choose $\overline{u}$?

$$\begin{aligned} -u'' &= 1 \quad \text{on } (-1,0) \\ u(-1) &= 0 \\ u(0) &= \overline{u} \end{aligned} \qquad \begin{aligned} -u'' &= 1 \quad \text{on } (0,1) \\ u(0) &= \overline{u} \\ u(1) &= 0 \end{aligned}$$

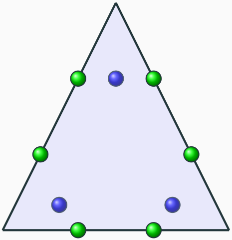


Continuous flux

Equation for $\overline{u}$: to ensure a continuous flux

HDG Degrees of Freedom

$k = 1$



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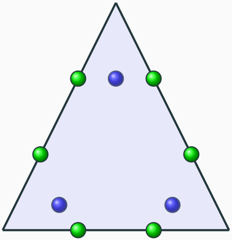
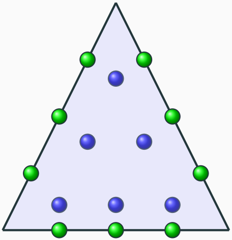
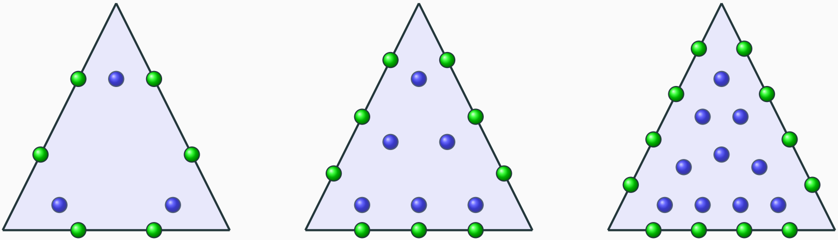


$k = 2$



HDG Degrees of Freedom

$k = 1$ $k = 2$ $k = 3$



Linear problem

Weak form

Seek $(u_h, \overline{u}_h) \in V_h \times \overline{V}_h$ such that forall $(v_h, \overline{v}_h) \in V_h \times \overline{V}_h$

$$a_{HDG}((u_h, \overline{u}_h), (v_h, \overline{v}_h)) = l_{HDG}(v_h, \overline{v}_h)$$

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Linear system System form

$$a_{II}(u_h, v_h) + a_{FI}(\overline{u}_h, v_h) = l_I(v_h)$$
$$a_{IF}(u_h, \overline{v}_h) + a_{FF}(\overline{u}_h, \overline{v}_h) = l_F(\overline{v}_h)$$

Linear problem

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Seek $(u_h, \overline{u}_h) \in V_h \times \overline{V}_h$ such that forall $(v_h, \overline{v}_h) \in V_h \times \overline{V}_h$

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Block system

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} U \\ \overline{U} \end{bmatrix} = \begin{bmatrix} F \\ G \end{bmatrix}$$

Schur-complement

A is block diagonal

$$\begin{aligned} AU + B\overline{U} &= F \\ CU + D\overline{U} &= G \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} U &= A^{-1}(F - B\overline{U}) \\ CA^{-1}(F - B\overline{U}) + D\overline{U} &= G \end{aligned}$$

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Solution in two steps

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Solution in two steps

$$(D - CA^{-1}B)\overline{U} = G - CA^{-1}F$$

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Solution in two steps

$$(D - CA^{-1}B)\overline{U} = G - CA^{-1}F$$

$$U = A^{-1}(F - B\overline{U})$$

Diffusion Dominated Problem

Consider the advection-diffusion problem

$$\begin{aligned} -\kappa \Delta u + \vec{c} \cdot \nabla u &= f \qquad \text{in } \Omega = [0,1] \times [0,1], \\ u &= g_D \qquad \text{on } \Gamma = \partial\Omega, \end{aligned}$$

with exact solution $u(x,y) = \sin(6x)\sin(6y)$, f and g_D are derived from this exact solution, $\vec{c} = (-1,1)^T$ and κ is the diffusion coefficient.

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Convergence rates

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch^{k+1}$$

$\kappa = 1$

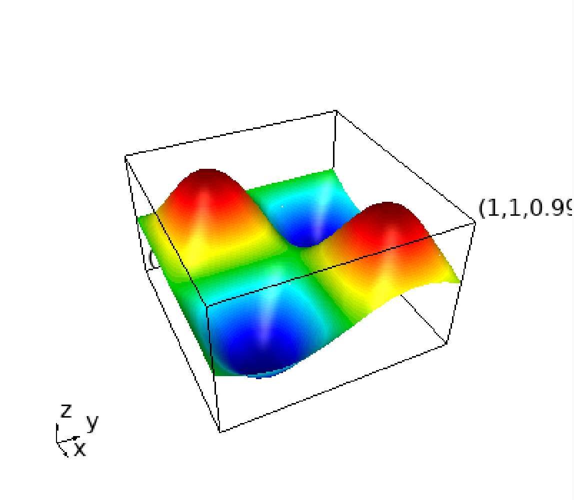

Figure 5: $||u_{HDG} - u_e||_{L_2} = 3.7621e - 4$

Pure Advection Problem

Consider the same problem

$$\begin{aligned} -\kappa \Delta u + \vec{c} \cdot \nabla u &= f \qquad \text{in } \Omega = [0,1] \times [0,1], \\ u &= g_D \qquad \text{on } \Gamma = \partial\Omega, \end{aligned}$$

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$\kappa = 0$

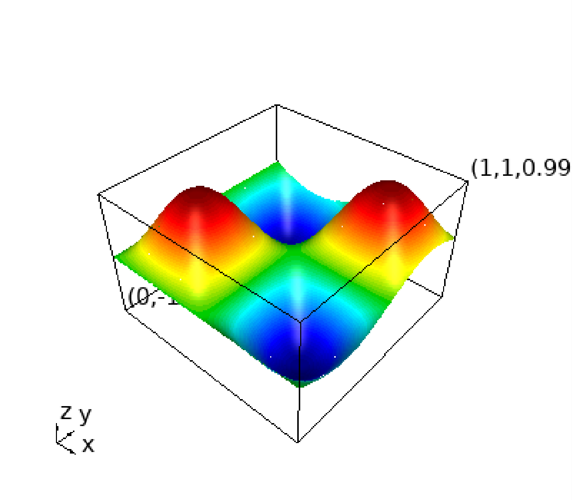
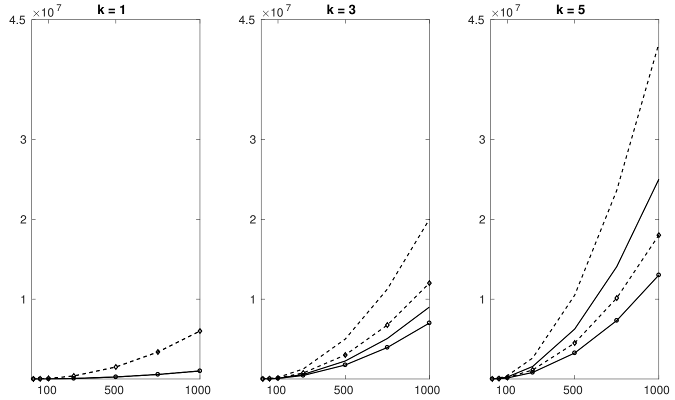


Figure 6: $||u_{HDG} - u_e||_{L_2} = 3.0956e - 4$

Comparison of the degrees of freedom

$n \times n$ uniform structured triangular mesh



Degrees of freedom for polynomial degree $k = 1, 3, 5$.
Continuous line CG, dashed line DG, Continuous line with circles EDG, dashed line with diamonds HDG

Matrix Sizes Demonstration

Consider the Poisson problem

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega = [0,1] \times [0,1] \\ u &= g_D \qquad \text{on } \Gamma = \partial\Omega. \end{aligned}$$

We are going to use the same mesh for all the discretizations.

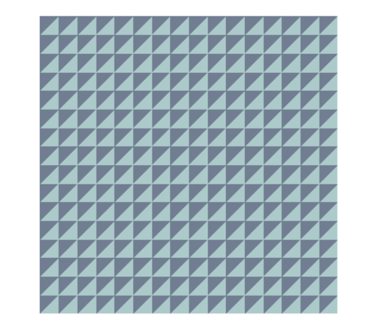
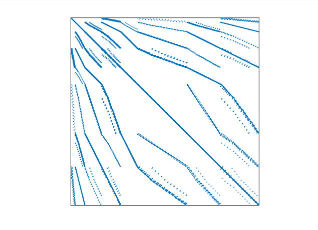
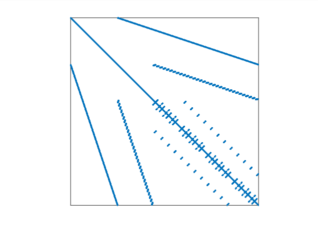


Figure 7: The mesh

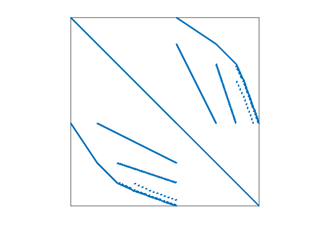
Matrix Properties $k = 2$



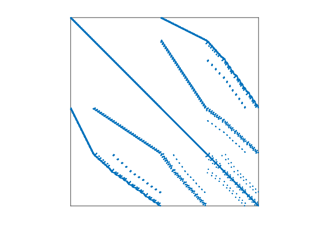
(a) CG: n=1089, nnz=8961



(b) DG: n=3072, nnz=71424



(c) HDG: n=5472, nnz=76128



(d) SC: n=2400, nnz=34848

Matrix Sizes $k = 5$

Table 1: Matrix size(n) and #Nonzeros(nnz) for different discretizations of order 5

	n	nnz
CG	6561	199521
DG	10752	874944
HDG	15552	617664
SC	<u>4800</u>	139392

Matrix Sizes $k = 5$

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- Linear system size is smaller than DG, and CG if $k \geq 4$.

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IPDG-H for ADR Problems

General equation

Consider the general advection-diffusion-reaction problem

$$
\begin{aligned}
\nabla \cdot (-\kappa \nabla u + \vec{b} u) + cu &= f \qquad \text{in } \Omega, \\
u &= g_D \qquad \text{on } \Gamma = \partial\Omega.
\end{aligned}
$$

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Rewrite it in mixed form, let $q = -\kappa \nabla u$;

$$\begin{aligned} q + \kappa \nabla u = 0 \qquad & \text{in } \Omega, \\ \nabla \cdot (q + \vec{b} u) + cu = f \qquad & \text{in } \Omega, \\ u = g_D \qquad & \text{on } \Gamma = \partial\Omega. \end{aligned}$$

Start by meshing the domain Ω; $\mathcal{T} = \{K\}$, non-overlapping elements, and,

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Assumption; $F \in \mathcal{F}$ has nonzero $(d - 1)$ Lebesgue measure, where d is the dimensionality of Ω.

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$(\cdot,\cdot)_K$: standart $L^2(K)$-inner product

$< \cdot,\cdot >_F$: standart $L^2(F)$-inner product

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$(\cdot,\cdot)_K$: standart $L^2(K)$-inner product

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$(\cdot,\cdot)_\Omega = \sum_{K\in\mathcal{T}} (\cdot,\cdot)_K$

$< \cdot,\cdot >_{\partial\Omega} = \sum_{F\in\mathcal{F}} < \cdot,\cdot >_F$

Now, define the spaces,

$$
\begin{aligned}
R_h &= \{r_h \in \left[L^2(\Omega)\right]^d, r_h \in [P_k(K)]^d \quad \forall K \in \mathcal{T}\} \\
V_h &= \{V_h \in L^2(\Omega), v_h \in P_k(K) \quad \forall K \in \mathcal{T}\}
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\end{aligned}
$$

and multiply by test functions r, v over Ω, and integrate,

$$
\begin{aligned}
(q, r)_\Omega + (\kappa \nabla u, r)_\Omega &= 0 \\
(\nabla \cdot (q + \vec{b}u), v)_\Omega + (cu, v)_\Omega &= (f, v)_\Omega.
\end{aligned}
$$

Project the boundary conditions to boundary faces and enforce them strongly

Now apply integration by parts wherever it is necessary,

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From first line,

$$
\begin{aligned}
(q, r)_\Omega &= (u, \kappa \nabla \cdot r)_\Omega - < \widehat{u}, \kappa r \cdot n >_{\partial\Omega} \\
&= -(\kappa \nabla u, r)_\Omega + < u - \widehat{u}, \kappa r \cdot n >_{\partial\Omega} .
\end{aligned}
$$

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$$\begin{aligned}(q, r)_\Omega &= (u, \kappa \nabla \cdot r)_\Omega - < \widehat{u}, \kappa r \cdot n >_{\partial\Omega} \\ &= -(\kappa \nabla u, r)_\Omega + < u - \widehat{u}, \kappa r \cdot n >_{\partial\Omega} .\end{aligned}$$

Second line is longer, consists more terms, hard to keep it tidy,

$$\begin{aligned}&-(\vec{b}u, \nabla v)_\Omega + < \widehat{\vec{b}u \cdot \vec{n}}, v >_{\partial\Omega} \\ &-(q, \nabla v)_\Omega + < \hat{q} \cdot \vec{n}, v >_{\partial\Omega} + (cu, v)_\Omega = (f, v)_\Omega.\end{aligned}$$

To reduce the number of these equations, pick $r = \nabla v$ and substitute $(q, r)_\Omega$ for $(q, \nabla v)_\Omega$

$$-(\vec{b}u, \nabla v)_\Omega + < \widehat{\vec{b}u \cdot \vec{n}}, v >_{\partial\Omega} + (\kappa \nabla u, \nabla v)_\Omega$$
$$- < u - \widehat{u}, \kappa \nabla v \cdot n >_{\partial\Omega} + < \hat{q} \cdot \vec{n}, v >_{\partial\Omega} + (cu, v)_\Omega = (f, v)_\Omega.$$

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&- < u - \widehat{u}, \kappa \nabla v \cdot n >_{\partial\Omega} + < \hat{q} \cdot \vec{n}, v >_{\partial\Omega} + (cu, v)_\Omega = (f, v)_\Omega .
\end{aligned}
$$

It might be desirable to keep the mixed form sometimes, i.e. for superconvergent methods with diffusion dominated problems.

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\end{aligned}
$$

It might be desirable to keep the mixed form sometimes, i.e. for superconvergent methods with diffusion dominated problems.

Introduce $\lambda \in M_h$, where,

$$
M_h = \{\mu_h \in L^2(\mathcal{F}), \mu_h \in P_k(F) \quad \forall F \in \mathcal{F}\}
$$

which is a function that only exists on the faces of the elements.

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$$\widehat{\vec{b}u \cdot \vec{n}} = \vec{b}u \cdot \vec{n} + \zeta \vec{b} \cdot \vec{n} (\lambda - u) = (1 - \zeta)\, \vec{b}u \cdot \vec{n} + \zeta \vec{b} \cdot \vec{n} \lambda,$$
$$\widehat{u} = \lambda,$$
$$\hat{q} = -\kappa \nabla u - \frac{\alpha}{h_K} \kappa \vec{n}(\lambda - u),$$

where ζ is an indicator function for interelement boundary (1 for inflow, 0 for outflow).

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2 unknowns: λ and u, 1 equation! Enforce continuity of the fluxes

through faces;

$$\left(< \widehat{\vec{b}u\cdot\vec{n}}, \mu >_{\partial\Omega} + < \hat{q}\cdot\vec{n}, \mu >_{\partial\Omega}\right) = 0.$$

Weak formulation

Find $(u, \lambda) \in V_h \times M_h$ s.t. $\forall (v, \mu) \in V_h \times M_h$,

$$-(\vec{b}u, \nabla v)_\Omega + < \widehat{\vec{b}u \cdot \vec{n}}, v >_{\partial\Omega} + (\kappa \nabla u, \nabla v)_\Omega$$
$$- < u - \widehat{u}, \kappa \nabla v \cdot n >_{\partial\Omega} + < \hat{q} \cdot \vec{n}, v >_{\partial\Omega} + (cu, v)_\Omega = (f, v)_\Omega,$$

and,

$$-\left(< \widehat{\vec{b}u \cdot \vec{n}}, \mu >_{\partial\Omega} + < \hat{q} \cdot \vec{n}, \mu >_{\partial\Omega} \right) = 0.$$

Block structure

Contents of each block,

$$\begin{bmatrix} [0] & [1] \\ [2] & [3] \end{bmatrix} = \begin{bmatrix} (u, v) & (\lambda, v) \\ (u, \mu) & (\lambda, \mu) \end{bmatrix}.$$

Reminder: First block is block diagonal, so Schur complement of this system is easy to compute.

Advantages

- Smaller linear system to solve
- Usually more accurate
- Better conditioned

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Better for fluid dynamics problems;

- H(div)-conforming spaces
- Exactly pointwise divergence free velocity fields (incompressibility)
- Mass conservation
- Momentum conservation
- Energy stability (transient problems)

IPDG-H for the Stokes Problem

The Stokes Problem

Given $I = (t_0, t_f]$, $f : \Omega \times I \to \mathbb{R}^d$ and $u_0 = \Omega \times t_0 \to \mathbb{R}^d$, the Stokes problem for $u : \Omega \times I \to \mathbb{R}^d$ is

$$
\begin{aligned}
\partial_t u + \nabla \cdot \sigma &= f \quad \text{in } \Omega, \\
\nabla \cdot u &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \Gamma = \partial\Omega, \\
\int_\Omega p \mathrm{d}x &= 0,
\end{aligned}
$$

where $\sigma = p\mathbb{I} - \nabla u$.

Define the spaces,

$$
\begin{aligned}
V_h &= \{v_h \in \left[L^2(\mathcal{T})\right]^d, v_h \in [P_k(K)]^d \quad \forall K \in \mathcal{T}\} \\
\bar{V}_h &= \{\bar{v}_h \in \left[L^2(\mathcal{F})\right]^d, \bar{v}_h \in [P_k(F)]^d \quad \forall F \in \mathcal{F}\} \\
Q_h &= \{q_h \in L^2(\mathcal{T}), q_h \in P_{k-1}(K) \quad \forall K \in \mathcal{T}\} \\
\bar{Q}_h &= \{\bar{q}_h \in L^2(\mathcal{F}), \bar{q}_h \in P_k(F) \quad \forall F \in \mathcal{F}\}
\end{aligned}
$$

Weak formulation

Find $(u, \bar{u}, p, \bar{p}) \in V_h \times \bar{V}_h \times Q_h \times \bar{Q}_h$ s.t.
$\forall (v, \bar{v}, q, \bar{q}) \in V_h \times \bar{V}_h \times Q_h \times \bar{Q}_h$,

$$\sum_{K \in \mathcal{T}} \int_K \nabla u : \nabla v \, \mathrm{d}x + \sum_{K \in \mathcal{T}} \int_{\partial K} (\bar{u} - u) \cdot \frac{\partial v}{\partial n} \, \mathrm{d}s - \sum_{K \in \mathcal{T}} \int_K p \nabla \cdot v \, \mathrm{d}x + \sum_{K \in \mathcal{T}} \int_{\partial K} \hat{\sigma} n \cdot (v - \bar{v}) \cdot \, \mathrm{d}s = \sum_{K \in \mathcal{T}} \int_K f \cdot v \, \mathrm{d}x$$

and

$$\sum_{K \in \mathcal{T}} \int_K u \cdot \nabla q \, \mathrm{d}x + \sum_{K \in \mathcal{T}} \int_{\partial K} \hat{u} \cdot n(\bar{q} - q) \, \mathrm{d}s - \int_\Gamma \bar{u} \cdot n \bar{q} \, \mathrm{d}s = 0.$$

Numerical Fluxes

$$
\begin{aligned}
\hat{\sigma} &= -\nabla u + \bar{p} I - \frac{\alpha_v}{h_K}(\bar{u} - u) \otimes n, \\
\hat{u} &= u - \alpha_p h_K (\bar{p} - p) n.
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α_p can be set to zero.

Some Insights to Weak Formulation

$$\sum_{K\in\mathcal{T}}\int_K \nabla u : \nabla v \ \mathrm{d}x + \sum_{K\in\mathcal{T}}\int_{\partial K}(\bar{u}-u)\cdot\frac{\partial v}{\partial n}\ \mathrm{d}s - \sum_{K\in\mathcal{T}}\int_K p\nabla\cdot v \ \mathrm{d}x$$
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Global energy stability: $\frac{d}{dt} \int_K |u|^2 \, \mathrm{d}x \leq 0$.