Introduction to Hybridized Discontinuous Galerkin (HDG) Methods

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(Not so) Brief Overview
Steady state problem

General second order PDE

\[ u \in C^2(\Omega) \cap C^1(\overline{\Omega}) \ (\Omega \subset \mathbb{R}^d, \ d \geq 1 \text{ bounded open domain}) \]

\[-\nabla \cdot \nu \nabla u + \nabla \cdot (b u) = f \quad \text{in } \Omega, \]

\[ u = 0 \quad \text{on } \Gamma. \]
Steady state problem

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Possible discretizations
Steady state problem

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Possible discretizations

1. Finite Difference Method: use \( \partial_x u(x_i) \approx \frac{u(x_i+h)-u(x_i)}{h} \) with "fixed" \( h \)
Steady state problem

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2. Finite Volume Method: use the integral form of the equation
Steady state problem

**General second order PDE**

\[ u \in C^2(\Omega) \cap C^1(\overline{\Omega}) \ (\Omega \subset \mathbb{R}^d, \ d \geq 1 \text{ bounded open domain}) \]

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**Possible discretizations**

1. Finite Difference Method: use \( \partial_x u(x_i) \approx \frac{u(x_i+h)-u(x_i)}{h} \) with "fixed" \( h \)

2. Finite Volume Method: use the integral form of the equation

3. Finite Element Method: use the weak form
FDM

- easy to implement
FDM

- easy to implement
- higher order is complicated
FDM

- easy to implement
- higher order is complicated
- complicated for complex geometry or unstructured mesh
FDM

- easy to implement
- higher order is complicated
- complicated for complex geometry or unstructured mesh
- not conservative
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**FVM**

- conservative discretization
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FVM

- conservative discretization
- higher order is complicated
- complicated for unstructured grid
- poor performance for convection dominated problems
- not conservative
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- poor performance for convection dominated problems
Weak form

\[-\nabla \cdot \nu \nabla u + \nabla \cdot (bu) = f \quad \text{in } \Omega\]

multiply by \(v\) and IBP

\[\int_{\Omega} \nu \nabla u \cdot \nabla v - \int_{\Omega} ub \cdot \nabla v + BC = \int_{\Omega} f v\]

\[a(u, v) + l(v)\]
Weak form & Classical Galerkin

Weak form

\[-\nabla \cdot \nu \nabla u + \nabla \cdot (bu) = f \quad \text{in } \Omega\]

multiply by \(v\) and IBP

\[
\int_{\Omega} \nu \nabla u \cdot \nabla v - \int_{\Omega} u b \cdot \nabla v + BC = \int_{\Omega} f v
\]

\[
\begin{align*}
&\begin{array}{c}
a(u,v) \\
l(v)
\end{array}
\end{align*}
\]

Only the first derivative appears in the formula
Weak form & Classical Galerkin

Weak form

\[-\nabla \cdot \nu \nabla u + \nabla \cdot (bu) = f \quad \text{in } \Omega\]

multiply by \(v\) and IBP

\[
\int_{\Omega} \nu \nabla u \cdot \nabla v - \int_{\Omega} ub \cdot \nabla v + BC = \int_{\Omega} f v
\]

\[
a(u, v) = \int_{\Omega} f v = l(v)
\]

Only the first derivative appears in the formula

\(u, v \in V = H^1_0(\Omega) = \{v \in L^2(\Omega) : \nabla v \in [L^2(\Omega)]^d, \ v|_{\Gamma} = 0\}\)
\[ ?u \in V : a(u, v) = l(v) \quad \forall v \in V \]

- \( V \) is infinite dimensional
\[ ?u \in V : a(u, v) = l(v) \quad \forall v \in V \]

- \(V\) is infinite dimensional
- Restrict to a finite dimensional subspace \(V_h \subset V\)
?u_h \in V_h : a(u_h, v_h) = l(v_h) \quad \forall v_h \in V_h

- $V$ is infinite dimensional
- Restrict to a finite dimensional subspace $V_h \subset V$
- $\mathcal{T}_h$: mesh over $\Omega$, $V_h$: piecewise polynomials that are continuous
\[ u_h \in V_h : a(u_h, v_h) = l(v_h) \quad \forall v_h \in V_h \]

- \( V \) is infinite dimensional
- Restrict to a finite dimensional subspace \( V_h \subset V \)
- \( T_h \): mesh over \( \Omega \), \( V_h \): piecewise polynomials that are continuous

\[ V_h = \{ v \in L^2(\Omega) : v \in P_k(K), \forall K \in T_h \} \cap C(\Omega) \]
\( \forall \nu_h \in V_h \) : 
\( a(u_h, \nu_h) = l(\nu_h) \) 

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- Restrict to a finite dimensional subspace \( V_h \subset V \)
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\[ V_h = \{ v \in L^2(\Omega) : v \in P_k(K), \forall K \in T_h \} \cap C(\Omega) \]

- Basis with compact support

\( u_h \in V_h \) : 
\( a(u_h, \nu_h) = l(\nu_h) \) 
\( \forall \nu_h \in V_h \)
\[ u_h \in V_h : a(u_h, v_h) = l(v_h) \quad \forall v_h \in V_h \]

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- Basis with compact support
- Easy to integrate
\[ \tilde{u}_h \in V_h : a(u_h, v_h) = l(v_h) \quad \forall v_h \in V_h \]

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\[ V_h = \{ v \in L^2(\Omega) : v \in P_k(K), \forall K \in T_h \} \cap C(\Omega) \]

- Basis with compact support
- Easy to integrate
- Easy to go for high polynomial degree
Choose a basis of $V_h : \{ \Phi_i, \cdots, \Phi_N \}$
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Seek the coefficients $\{ c_i \}$ such that $u_h = \sum_{i=1}^{N} c_i \Phi_i$
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**Linear system**

$A c = b$

where

- $A_{i,j} = a(\Phi_j, \Phi_i)$
Linear system

Choose a basis of $V_h : \{\Phi_i, \cdots, \Phi_N\}$

Seek the coefficients $\{c_i\}$ such that $u_h = \sum_{i=1}^{N} c_i \Phi_i$

**Linear system**

$$Ac = b$$

where

- $A_{i,j} = a(\Phi_j, \Phi_i)$
- $b_i = l(\Phi_i)$
Choose a basis of $V_h : \{\Phi_i, \cdots, \Phi_N\}$

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**Linear system**

$$Ac = b$$

where

- $A_{i,j} = a(\Phi_j, \Phi_i)$
- $b_i = l(\Phi_i)$
- $A$ is very sparse
Choose a basis of $V_h : \{ \Phi_i, \cdots, \Phi_N \}$

Seek the coefficients $\{ c_i \}$ such that $u_h = \sum_{i=1}^{N} c_i \Phi_i$

**Linear system**

$$A c = b$$

where

- $A_{i,j} = a(\Phi_j, \Phi_i)$
- $b_i = l(\Phi_i)$
- $A$ is very sparse
- Direct or iterative solver?
Choose a basis of $V_h : \{\Phi_i, \cdots, \Phi_N\}$

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**Linear system**

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where

- $A_{i,j} = a(\Phi_j, \Phi_i)$
- $b_i = l(\Phi_i)$
- $A$ is very sparse
- Direct or iterative solver?
- Size vs condition number
Degrees of Freedoms in 2D

\[ k = 1 \]
Degrees of Freddoms in 2D

\[ k = 1 \]

\[ k = 2 \]
Degrees of Freedoms in 2D

\( k = 1 \)

\( k = 2 \)

\( k = 3 \)
Consider the advection-diffusion problem

\[-\kappa \Delta u + \vec{c} \cdot \nabla u = f \quad \text{in } \Omega = [0, 1] \times [0, 1],
\]

\[u = g_D \quad \text{on } \Gamma = \partial \Omega,
\]

with exact solution \( u(x, y) = \sin(6x) \sin(6y) \), \( f \) and \( g_D \) are derived from this exact solution, \( \vec{c} = (-1, 1)^T \) and \( \kappa \) is the diffusion coefficient.
Consider the advection-diffusion problem

\[-\kappa \Delta u + \vec{c} \cdot \nabla u = f \quad \text{in } \Omega = [0, 1] \times [0, 1],\]
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**Convergence rates**

\[\|u - u_h\|_{L^2(\Omega)} \leq C h^{k+1}\]
\( \kappa = 1 \)

**Figure 1:** \( \| \mathbf{u}_{CG} - \mathbf{u}_e \|_{L_2} = 4.7704 \times 10^{-4} \)
Consider the same problem

$$\begin{align*}
-\kappa \Delta u + \vec{c} \cdot \nabla u &= f \quad \text{in } \Omega = [0, 1] \times [0, 1], \\
u &= g_D \quad \text{on } \Gamma = \partial\Omega,
\end{align*}$$

with exact solution $u(x, y) = \sin(6x) \sin(6y)$, $f$ and $g_D$ are derived from this exact solution, $\vec{c} = (-1, 1)^T$ and $\kappa = 0$. 
Figure 2: $\|u_{CG} - u_e\|_2 = 7.02071$
DG
Possible improvement: DG

FVM

- numerical fluxes over the elements
- upwind flux
Possible improvement: DG

FVM
- numerical fluxes over the elements
- upwind flux

CG
- higher order discretization
Possible improvement: DG

**FVM**
- numerical fluxes over the elements
- upwind flux

**DG**
- derive weak form starting from one element
- connection between elements via fluxes
- higher order discretization

**CG**
- higher order discretization
Mesh first
Mesh first

Rewrite $-\nabla \cdot \nu \nabla u + \nabla \cdot (bu) = f$ using $q = -\nabla u$

First order system

$$\nu \nabla \cdot q + \nabla \cdot (bu) = f$$

$$q + \nabla u = 0$$
Mesh first

Rewrite $-\nabla \cdot \nu \nabla u + \nabla \cdot (bu) = f$ using $q = -\nabla u$

First order system

$$\nu \nabla \cdot q + \nabla \cdot (bu) = f$$
$$q + \nabla u = 0$$

IBP on mesh element $K +$ discretization

$$\int_K fw_h = - \int_K \nu q_h \cdot \nabla w_h + \int_{\partial K} \nu \hat{q}_h \cdot n w_h$$
$$- \int_K u_h b \cdot \nabla w_h + \int_{\partial K} w_h \hat{u}_h b \cdot n$$
$$\int_K q_h \cdot v_h = - \int_K \nabla u_h \cdot v_h + \int_{\partial K} (u_h - \hat{u}_h) v_h \cdot n$$
If \( \mathbf{v} = \nu \nabla w_h \)

\[
\int_K f w_h = \int_K \nu \nabla u_h \cdot \nabla w_h + \int_{\partial K} (\hat{u}_h - u_h) \nu \nabla w_h \cdot \mathbf{n} + \int_{\partial K} \nu \hat{q}_h \cdot \mathbf{n} w_h \\
- \int_K u_h \mathbf{b} \cdot \nabla w_h + \int_{\partial K} w_h \hat{u}_h \mathbf{b} \cdot \mathbf{n}
\]
If \( \mathbf{v} = \nu \nabla w_h \)

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- \int_K u_h \mathbf{b} \cdot \nabla w_h + \int_{\partial K} w_h \hat{u}_h \mathbf{b} \cdot \mathbf{n}
\]

Summing over all \( K \in \mathcal{T}_h \)
If \( v = \nu \nabla w_h \)

\[
\int_K f w_h = \int_K \nu \nabla u_h \cdot \nabla w_h + \int_{\partial K} (\hat{u}_h - u_h) \nu \nabla w_h \cdot \mathbf{n} + \int_{\partial K} \nu \hat{q}_h \cdot \mathbf{n} w_h \\
- \int_K u_h \mathbf{b} \cdot \nabla w_h + \int_{\partial K} w_h \hat{u}_h \mathbf{b} \cdot \mathbf{n}
\]

Summing over all \( K \in \mathcal{T}_h \)

The interior faces will show up twice
Choice of the numerical flux

Advection
Use upwinding

\[
\hat{u}_h = \begin{cases} 
  u_L & \text{if } b \cdot n \geq 0 \\
  u_R & \text{if } b \cdot n < 0 
\end{cases}
\]
Choice of the numerical flux

**Advection**

Use upwinding

\[
\hat{u}_h = \begin{cases} 
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**Diffusion part**

Plenty of possibilities (see Brezzi-Marini survey)
Advection

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Diffusion part

Plenty of possibilities (see Brezzi-Marini survey)

Interior penalty: \( \hat{q}_h = \nabla u_L + \frac{\alpha}{h} [u_h] n = \nabla u_L + \frac{\alpha}{h} (u_L - u_R) n \)
Choice of the numerical flux

Advection
Use upwinding

\[ \hat{u}_h = \begin{cases} u_L & \text{if } b \cdot n \geq 0 \\ u_R & \text{if } b \cdot n < 0 \end{cases} \]

Diffusion part
Plenty of possibilities (see Brezzi-Marini survey)

Interior penalty: \( \hat{q}_h = \nabla u_L + \frac{\alpha}{h} [u_h] n = \nabla u_L + \frac{\alpha}{h} (u_L - u_R)n \)

Example with 2 equations: Local DG: \( \hat{q}_h = q_L + \tau (u_L - u_R)n \)
Upwind in 2D
Upwind in 2D
Upwind in 2D
Upwind in 2D

\[ u^2_R \quad b \quad u^1_L \quad u_L \]
IP approximate weak form

Seek $u_h \in V_h$ such that $a_{DG}(u_h, v_h) = l_{DG}(v_h)$ for all $v_h \in V_h$. 
IP approximate weak form

Seek $u_h \in V_h$ such that $a_{DG}(u_h, v_h) = l_{DG}(v_h)$ for all $v_h \in V_h$.

Looks like CG, but it is totally different
IP discretization

**IP approximate weak form**
Seek $u_h \in V_h$ such that $a_{DG}(u_h, v_h) = l_{DG}(v_h)$ for all $v_h \in V_h$.

Looks like CG, but it is totally different

**IP notations**

$$a_{DG}(u_h, v_h) = \sum_{K \in T_h} \int_{K} \nu \nabla u_h \cdot \nabla w_h - \sum_{K \in T_h} \int_{\partial K} u_h b \cdot \nabla v_h$$

+ interior face terms

$$l_{DG}(v_h) = \sum_{K \in T_h} \int_{K} f v_h + BC$$

$$V_h = \{ v \in L^2(\Omega) : v \in P_k(K), \forall K \in T_h \}$$
$k = 1$
2D DG basis functions

$k = 1$

$k = 2$
2D DG basis functions

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## CG vs DG pros and cons

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<th>DG</th>
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<td>• lower number of degrees of freedom</td>
<td>• higher number of degrees of freedom</td>
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- **CG**
  - Lower number of degrees of freedom
  - Diffusion dominated cases are easier to solve by iterative methods
  - Fails for convection dominated cases
  - Hard to do hp adaptivity: the unknowns on different elements are connected

- **DG**
  - Higher number of degrees of freedom
  - Harder to solve with an iterative solver
  - Works better for convection dominated cases
  - Hp adaptivity is easy: the unknowns on different elements are not connected
CG vs DG pros and cons

**CG**
- lower number of degrees of freedom
- diffusion dominated cases are easier to solve by iterative methods

**DG**
- higher number of degrees of freedom
- harder to solve with an iterative solver
CG vs DG pros and cons

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CG vs DG pros and cons

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- diffusion dominated cases are easier to solve by iterative methods
- fails for convection dominated cases
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- $hp$ adaptivity is easy: the unknowns on different elements are not connected
Consider the advection-diffusion problem

\[-\kappa \Delta u + \vec{c} \cdot \nabla u = f \quad \text{in } \Omega = [0, 1] \times [0, 1],\]
\[u = g_D \quad \text{on } \Gamma = \partial \Omega,\]

with exact solution \(u(x, y) = \sin(6x) \sin(6y)\), \(f\) and \(g_D\) are derived from this exact solution, \(\vec{c} = (-1, 1)^T\) and \(\kappa\) is the diffusion coefficient. 
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**Convergence rates**

\[\|u - u_h\|_{L^2(\Omega)} \leq Ch^{k+1}\]
$\kappa = 1$

**Figure 3:** $\|u_{DG} - u_e\|_{L_2} = 3.8546e - 4$
Consider the same problem

\[-\kappa \Delta u + \vec{c} \cdot \nabla u = f \quad \text{in } \Omega = [0, 1] \times [0, 1],\]
\[u = g_D \quad \text{on } \Gamma = \partial \Omega,\]

with exact solution \( u(x, y) = \sin(6x) \sin(6y) \), \( f \) and \( g_D \) are derived from this exact solution, \( \vec{c} = (-1, 1)^T \) and \( \kappa = 0 \).
Figure 4: $\|u_{DG} - u_e\|_{L^2} = 3.0956 \times 10^{-4}$
Idea of HDG
DG

- derive weak form on one element
Hybridizable DG

DG

- derive weak form on one element
- connection between elements via fluxes
Hybridizable DG

DG
- derive weak form on one element
- connection between elements via fluxes

HDG
- derive weak form on one element
Hybridizable DG

DG
- derive weak form on one element
- connection between elements via fluxes

HDG
- derive weak form on one element
- additional unknowns on the edges
Hybridizable DG

DG

- derive weak form on one element
- connection between elements via fluxes

HDG

- derive weak form on one element
- additional unknowns on the edges
- connection between elements via fluxes that uses functions on the edges
\[ u_h \in V_h = \{ v_h \in L^2(\Omega), \, v_h \in P_k(K) \, \forall K \in \mathcal{T}_h \} \]

\[ \overline{u}_h \in \overline{V}_h = \{ \overline{v}_h \in L^2(\mathcal{F}), \, \overline{v}_h \in P_k(F) \, \forall F \in \mathcal{F} \} \]
DG fluxes

Advection: upwinding
DG fluxes

Advection: upwinding

Diffusion Interior penalty or Local DG or one of the many
HDG fluxes

DG fluxes
Advection: upwinding

Diffusion Interior penalty or Local DG or one of the many

HDG fluxes
Advection: \( \hat{u}_h = \begin{cases} u_L & \text{if } b \cdot n \geq 0 \\ \bar{u} & \text{if } b \cdot n < 0 \end{cases} \)
DG fluxes

Advection: upwinding

Diffusion Interior penalty or Local DG or one of the many

HDG fluxes

Advection: \( \hat{u}_h = \begin{cases} u_L & \text{if } b \cdot n \geq 0 \\ \bar{u} & \text{if } b \cdot n < 0 \end{cases} \)

Diffusion IP: \( \hat{q}_h = \nabla u_L + \frac{\alpha}{h} (u_L - \bar{u})n \)
HDG fluxes

DG fluxes
Advection: upwinding

Diffusion Interior penalty or Local DG or one of the many

HDG fluxes

Advection: \( \hat{u}_h = \begin{cases} u_L & \text{if } b \cdot n \geq 0 \\ \overline{u} & \text{if } b \cdot n < 0 \end{cases} \)

Diffusion IP: \( \hat{q}_h = \nabla u_L + \frac{\alpha}{h}(u_L - \overline{u})n \)

Local DG: \( \hat{q}_h = q_L + \tau(u_L - \overline{u})n \)
Upwind for HDG

$u_L$
Upwind for HDG
Upwind for HDG
Upwind for HDG

\[ \overline{u}^1 \]

\[ \overline{u}^2 \]

\[ u_L \]

\[ b \]

\[ u_L \]

\[ b \]
Solve $-u'' = 1$, $u(-1) = u(1) = 0$ as

**How to choose $\overline{u}$?**

$-u'' = 1$ on $(-1, 0)$  
$u(-1) = 0$  
$u(0) = \overline{u}$

$-u'' = 1$ on $(0, 1)$  
$u(0) = \overline{u}$  
$u(1) = 0$
Solve \(-u'' = 1, u(-1) = u(1) = 0\) as

**How to choose \(\bar{u}\)?**

\[
\begin{align*}
-u'' &= 1 \quad \text{on } (-1, 0) \\
\quad &\text{and} \\
-u'' &= 1 \quad \text{on } (0, 1) \\
u(-1) &= 0 \\
u(0) &= \bar{u} \\
u(1) &= 0
\end{align*}
\]

\[
\begin{align*}
\bar{u} &= 0.25 \\
\bar{u} &= 0.5 \\
\bar{u} &= 0.75
\end{align*}
\]
Solve \(-u'' = 1, u(-1) = u(1) = 0\) as

**How to choose \(\overline{u}\)?**

\[-u'' = 1 \quad \text{on } (-1, 0) \quad -u'' = 1 \quad \text{on } (0, 1)\]

\[u(-1) = 0 \quad \quad \quad \quad \quad u(0) = \overline{u} \quad \quad \quad \quad u(1) = 0\]

\(\overline{u} = 0.25\) \hspace{1cm} \(\overline{u} = 0.5\) \hspace{1cm} \(\overline{u} = 0.75\)

**Continuous flux**

Equation for \(\overline{u}\): to ensure a continuous flux
HDG Degrees of Freedom

\[ k = 1 \]
HDG Degrees of Freedom

$k = 1$

$k = 2$
HDG Degrees of Freedom

\[ k = 1 \quad k = 2 \quad k = 3 \]
Linear problem

Weak form

Seek \((u_h, \bar{u}_h) \in V_h \times \bar{V}_h\) such that forall \((v_h, \bar{v}_h) \in V_h \times \bar{V}_h\)

\[ a_{HDG}((u_h, \bar{u}_h), (v_h, \bar{v}_h)) = l_{HDG}(v_h, \bar{v}_h) \]
Linear problem

Weak form
Seek \((u_h, \bar{u}_h) \in V_h \times \bar{V}_h\) such that for all \((v_h, \bar{v}_h) \in V_h \times \bar{V}_h\)

\[
a_{HDG}((u_h, \bar{u}_h), (v_h, \bar{v}_h)) = l_{HDG}(v_h, \bar{v}_h)
\]

Linear system System form

\[
a_{II}(u_h, v_h) + a_{FI}(\bar{u}_h, v_h) = l_I(v_h)
\]
\[
a_{IF}(u_h, \bar{v}_h) + a_{FF}(\bar{u}_h, \bar{v}_h) = l_F(\bar{v}_h)
\]
Linear problem

Weak form
Seek \((u_h, \bar{u}_h) \in V_h \times \bar{V}_h\) such that for all \((v_h, \bar{v}_h) \in V_h \times \bar{V}_h\)

\[ a_{\text{HDG}}((u_h, \bar{u}_h), (v_h, \bar{v}_h)) = l_{\text{HDG}}(v_h, \bar{v}_h) \]

Linear system System form

\[ a_{\text{II}}(u_h, v_h) + a_{\text{FI}}(\bar{u}_h, v_h) = l_{\text{I}}(v_h) \]
\[ a_{\text{IF}}(u_h, \bar{v}_h) + a_{\text{FF}}(\bar{u}_h, \bar{v}_h) = l_{\text{F}}(\bar{v}_h) \]

Block system

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
U \\
\bar{U}
\end{bmatrix} =
\begin{bmatrix}
F \\
G
\end{bmatrix}
\]
Schur-complement

A is block diagonal

$$AU + B\overline{U} = F \quad \iff \quad U = A^{-1}(F - B\overline{U})$$

$$CU + D\overline{U} = G \quad \iff \quad CA^{-1}(F - B\overline{U}) + D\overline{U} = G$$
A is block diagonal

\[ AU + B\overline{U} = F \quad \Leftrightarrow \quad U = A^{-1}(F - B\overline{U}) \]

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Solution in two steps
Schur-complement

$A$ is block diagonal

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AU + B\bar{U} = F \quad \Leftrightarrow \quad U = A^{-1}(F - B\bar{U})
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\[
CU + D\bar{U} = G \quad \Leftrightarrow \quad CA^{-1}(F - B\bar{U}) + D\bar{U} = G
\]

Solution in two steps

\[
(D - CA^{-1}B)\bar{U} = G - CA^{-1}F
\]
Schur-complement

A is block diagonal

\[ AU + B\bar{U} = F \quad \iff \quad U = A^{-1}(F - B\bar{U}) \]
\[ CU + D\bar{U} = G \quad \iff \quad CA^{-1}(F - B\bar{U}) + D\bar{U} = G \]

Solution in two steps

\[(D - CA^{-1}B)\bar{U} = G - CA^{-1}F \]
\[ U = A^{-1}(F - B\bar{U}) \]
Consider the advection-diffusion problem

\[-\kappa \Delta u + \vec{c} \cdot \nabla u = f \quad \text{in } \Omega = [0, 1] \times [0, 1],\]

\[u = g_D \quad \text{on } \Gamma = \partial \Omega,\]

with exact solution \( u(x, y) = \sin(6x) \sin(6y) \), \( f \) and \( g_D \) are derived from this exact solution, \( \vec{c} = (-1, 1)^T \) and \( \kappa \) is the diffusion coefficient.
Consider the advection-diffusion problem

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**Convergence rates**

\[\|u - u_h\|_{L^2(\Omega)} \leq Ch^{k+1}\]
Figure 5: $\| u_{\text{HDG}} - u_e \|_{L^2} = 3.7621e^{-4}$
Consider the same problem

\[-\kappa \Delta u + \vec{c} \cdot \nabla u = f \quad \text{in } \Omega = [0, 1] \times [0, 1],\]

\[u = g_D \quad \text{on } \Gamma = \partial \Omega,\]

with exact solution \( u(x, y) = \sin(6x) \sin(6y), \) \( f \) and \( g_D \) are derived from this exact solution, \( \vec{c} = (-1, 1)^T \) and \( \kappa = 0. \)
\( \kappa = 0 \)

**Figure 6:** \( \|u_{\text{HDG}} - u_e\|_{L^2} = 3.0956 \times 10^{-4} \)
Comparison of the degrees of freedom

$n \times n$ uniform structured triangular mesh

Degrees of freedom for polynomial degree $k = 1, 3, 5$.
Continuous line CG, dashed line DG, Continuous line with circles EDG, dashed line with diamonds HDG
Consider the Poisson problem

$$-\Delta u = f \quad \text{in } \Omega = [0, 1] \times [0, 1]$$

$$u = g_D \quad \text{on } \Gamma = \partial \Omega.$$ 

We are going to use the same mesh for all the discretizations.
Figure 7: The mesh
Matrix Properties $k = 2$

(a) CG: $n=1089$, $\text{nnz}=8961$

(b) DG: $n=3072$, $\text{nnz}=71424$

(c) HDG: $n=5472$, $\text{nnz}=76128$

(d) SC: $n=2400$, $\text{nnz}=34848$
Matrix Sizes $k = 5$

Table 1: Matrix size($n$) and #Nonzeros(nnz) for different discretizations of order 5

<table>
<thead>
<tr>
<th></th>
<th>n</th>
<th>nnz</th>
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<tbody>
<tr>
<td>CG</td>
<td>6561</td>
<td>199521</td>
</tr>
<tr>
<td>DG</td>
<td>10752</td>
<td>874944</td>
</tr>
<tr>
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- Linear system size is smaller than DG, and CG if $k \geq 4$. 
Matrix Sizes $k = 5$

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- Linear system size is smaller than DG, and CG if $k \geq 4$.
- HDG is stable for advection-dominated flows.
IPDG-H for ADR Problems
Consider the general advection-diffusion-reaction problem

\[ \nabla \cdot (-\kappa \nabla u + \vec{b} u) + cu = f \quad \text{in } \Omega, \]
\[ u = g_D \quad \text{on } \Gamma = \partial \Omega. \]
Consider the general advection-diffusion-reaction problem
\[
\nabla \cdot (\kappa \nabla u + \vec{b} u) + cu = f \quad \text{in } \Omega,
\]
\[
u = g_D \quad \text{on } \Gamma = \partial \Omega.
\]

Rewrite it in mixed form, let \(q = -\kappa \nabla u\);
\[
q + \kappa \nabla u = 0 \quad \text{in } \Omega,
\]
\[
\nabla \cdot (q + \vec{b} u) + cu = f \quad \text{in } \Omega,
\]
\[
u = g_D \quad \text{on } \Gamma = \partial \Omega.
\]
Start by meshing the domain \( \Omega \); \( \mathcal{T} = \{K\} \), non-overlapping elements, and,
Start by meshing the domain $\Omega$; $\mathcal{T} = \{K\}$, non-overlapping elements, and,

$\mathcal{F}^i = \{F | F = \partial K^+ \cap \partial K^-\}$ and $\mathcal{F}^b = \{F | F = \partial K \cap \partial \Omega\}$, $\mathcal{F} = \mathcal{F}^i \cup \mathcal{F}^b$.

Assumption; $F \in \mathcal{F}$ has nonzero $(d - 1)$ Lebesgue measure, where $d$ is the dimensionality of $\Omega$. 
Start by meshing the domain $\Omega$; $\mathcal{T} = \{K\}$, non-overlapping elements, and,

$\mathcal{F}^i = \{F|F = \partial K^+ \cap \partial K^-\}$ and $\mathcal{F}^b = \{F|F = \partial K \cap \partial \Omega\}$,

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Assumption; $F \in \mathcal{F}$ has nonzero $(d - 1)$ Lebesgue measure, where $d$ is the dimensionality of $\Omega$.

$(\cdot, \cdot)_K$: standard $L^2(K)$-inner product

$<\cdot, \cdot>_F$: standard $L^2(F)$-inner product
Start by meshing the domain $\Omega$; $\mathcal{T} = \{ K \}$, non-overlapping elements, and,

$$\mathcal{F}^i = \{ F | F = \partial K^+ \cap \partial K^- \} \text{ and } \mathcal{F}^b = \{ F | F = \partial K \cap \partial \Omega \},$$

$$\mathcal{F} = \mathcal{F}^i \cup \mathcal{F}^b.$$

Assumption; $F \in \mathcal{F}$ has nonzero $(d - 1)$ Lebesgue measure, where $d$ is the dimensionality of $\Omega$.

$$(\cdot, \cdot)_K$:$ \text{ standard } L^2(K)$-inner product

$< \cdot, \cdot >_F$: $\text{ standard } L^2(F)$-inner product

$$(\cdot, \cdot)_\Omega = \sum_{K \in \mathcal{T}} (\cdot, \cdot)_K$$

$$< \cdot, \cdot >_{\partial \Omega} = \sum_{F \in \mathcal{F}} < \cdot, \cdot >_F$$
Now, define the spaces,

\[ R_h = \{ r_h \in [L^2(\Omega)]^d, r_h \in [P_k(K)]^d \quad \forall K \in \mathcal{T} \} \]
\[ V_h = \{ V_h \in L^2(\Omega), v_h \in P_k(K) \quad \forall K \in \mathcal{T} \} \]
Now, define the spaces,

\[
R_h = \{ r_h \in [L^2(\Omega)]^d, \; r_h \in [P_k(K)]^d \; \forall K \in \mathcal{T} \}
\]
\[
V_h = \{ V_h \in L^2(\Omega), \; v_h \in P_k(K) \; \forall K \in \mathcal{T} \}
\]

and multiply by test functions \( r, v \) over \( \Omega \), and integrate,

\[
(q, r)_\Omega + (\kappa \nabla u, r)_\Omega = 0
\]
\[
(\nabla \cdot (q + \vec{b}u), v)_\Omega + (cu, v)_\Omega = (f, v)_\Omega.
\]

Project the boundary conditions to boundary faces and enforce them strongly.
Now apply integration by parts wherever it is necessary,
Now apply integration by parts wherever it is necessary,

From first line,

\[
(q, r)_\Omega = (u, \kappa \nabla \cdot r)_\Omega - \langle \hat{u}, \kappa r \cdot n \rangle_{\partial \Omega} \\
= -(\kappa \nabla u, r)_\Omega + \langle u - \hat{u}, \kappa r \cdot n \rangle_{\partial \Omega}.
\]
Now apply integration by parts wherever it is necessary,

From first line,

\[ (q, r)_\Omega = (u, \kappa \nabla \cdot r)_\Omega - \langle \hat{u}, \kappa r \cdot n \rangle_{\partial \Omega} \]
\[ = -(\kappa \nabla u, r)_\Omega + \langle u - \hat{u}, \kappa r \cdot n \rangle_{\partial \Omega} . \]

Second line is longer, consists more terms, hard to keep it tidy,

\[ -(\vec{b}u, \nabla v)_\Omega + \langle \vec{b}u \cdot \vec{n}, v \rangle_{\partial \Omega} \]
\[ -(q, \nabla v)_\Omega + \langle \hat{q} \cdot \vec{n}, v \rangle_{\partial \Omega} + (cu, v)_\Omega = (f, v)_\Omega. \]
To reduce the number of these equations, pick \( r = \nabla v \) and substitute \((q, r)_{\Omega}\) for \((q, \nabla v)_{\Omega}\)

\[
- (\vec{b} u, \nabla v)_{\Omega} + < \vec{b} u \cdot \vec{n}, v >_{\partial \Omega} + (\kappa \nabla u, \nabla v)_{\Omega} \\
- < u - \hat{u}, \kappa \nabla v \cdot n >_{\partial \Omega} + < \hat{q} \cdot \vec{n}, v >_{\partial \Omega} + (c u, v)_{\Omega} = (f, v)_{\Omega}.
\]
To reduce the number of these equations, pick $r = \nabla v$ and substitute $(q, r)_\Omega$ for $(q, \nabla v)_\Omega$

$$-(\tilde{b}u, \nabla v)_\Omega + \langle \tilde{b}u \cdot \vec{n}, v \rangle_{\partial \Omega} + (\kappa \nabla u, \nabla v)_\Omega$$

$$- < u - \hat{u}, \kappa \nabla v \cdot n >_{\partial \Omega} + < \hat{q} \cdot \vec{n}, v >_{\partial \Omega} + (cu, v)_\Omega = (f, v)_\Omega.$$

It might be desirable to keep the mixed form sometimes, i.e. for superconvergent methods with diffusion dominated problems.
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\[
-(\vec{b}u, \nabla v)_\Omega + <\vec{b}u \cdot \vec{n}, v>_{\partial \Omega} + (\kappa \nabla u, \nabla v)_\Omega
- <u - \hat{u}, \kappa \nabla v \cdot \vec{n}>_{\partial \Omega} + <\hat{q} \cdot \vec{n}, v>_{\partial \Omega} + (cu, v)_\Omega = (f, v)_\Omega.
\]

It might be desirable to keep the mixed form sometimes, i.e. for superconvergent methods with diffusion dominated problems.

Introduce \( \lambda \in M_h \), where,

\[
M_h = \{ \mu_h \in L^2(\mathcal{F}), \mu_h \in P_k(F) \quad \forall F \in \mathcal{F} \}
\]

which is a function that only exists on the faces of the elements.
Define the fluxes using \( \lambda \), to get IP-HDG derivation,
Define the fluxes using $\lambda$, to get IP-HDG derivation,

$$
\hat{bu} \cdot \vec{n} = bu \cdot \vec{n} + \zeta \vec{b} \cdot \vec{n}(\lambda - u) = (1 - \zeta) \hat{bu} \cdot \vec{n} + \zeta \vec{b} \cdot \vec{n}\lambda,
$$

$$
\hat{u} = \lambda,
$$

$$
\hat{q} = -\kappa \nabla u - \frac{\alpha}{h_K} \kappa \vec{n}(\lambda - u),
$$

where $\zeta$ is an indicator function for interelement boundary (1 for inflow, 0 for outflow).
Define the fluxes using $\lambda$, to get IP-HDG derivation,

\[
\hat{b}u \cdot \hat{n} = b u \cdot n + \zeta b \cdot n (\lambda - u) = (1 - \zeta) b u \cdot n + \zeta b \cdot n \lambda,
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\[
\hat{q} = -\kappa \nabla u - \frac{\alpha}{h_K} \kappa \hat{n} (\lambda - u),
\]

where $\zeta$ is an indicator function for interelement boundary (1 for inflow, 0 for outflow).

2 unknowns: $\lambda$ and $u$, 1 equation!
Define the fluxes using $\lambda$, to get IP-HDG derivation,

$$
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$$

$$
\hat{u} = \lambda,
$$

$$
\hat{q} = -\kappa \nabla u - \frac{\alpha}{h_K} \kappa \vec{n}(\lambda - u),
$$

where $\zeta$ is an indicator function for interelement boundary (1 for inflow, 0 for outflow).

2 unknowns: $\lambda$ and $u$, 1 equation! Enforce continuity of the fluxes through faces;

$$
\left( < \hat{\vec{b}u} \cdot \vec{n}, \mu >_{\partial\Omega} + < \hat{q} \cdot \vec{n}, \mu >_{\partial\Omega} \right) = 0.
$$
Weak formulation

Find \((u, \lambda) \in V_h \times M_h\) s.t. \(\forall (v, \mu) \in V_h \times M_h\),

\[-(\tilde{b}u, \nabla v)_\Omega + <\tilde{b}u \cdot \vec{n}, v >_{\partial \Omega} + (\kappa \nabla u, \nabla v)_\Omega \]
\[\quad \quad - <u - \hat{u}, \kappa \nabla v \cdot \vec{n} >_{\partial \Omega} + <\hat{q} \cdot \vec{n}, v >_{\partial \Omega} + (c u, v)_\Omega = (f, v)_\Omega,\]

and,

\[-\left(<\tilde{b}u \cdot \vec{n}, \mu >_{\partial \Omega} + <\hat{q} \cdot \vec{n}, \mu >_{\partial \Omega}\right) = 0.\]
Contents of each block,

\[
\begin{bmatrix}
0 & 1 \\
2 & 3 \\
\end{bmatrix} = \begin{bmatrix}
(u, v) & (\lambda, v) \\
(u, \mu) & (\lambda, \mu) \\
\end{bmatrix}.
\]

Reminder: First block is block diagonal, so Schur complement of this system is easy to compute.
Advantages

- Smaller linear system to solve
- Usually more accurate
- Better conditioned
Advantages

• Smaller linear system to solve
• Usually more accurate
• Better conditioned

Better for fluid dynamics problems;

• $H(\text{div})$-conforming spaces
• Exactly pointwise divergence free velocity fields (incompressibility)
• Mass conservation
• Momentum conservation
• Energy stability (transient problems)
IPDG-H for the Stokes Problem
The Stokes Problem

Given $I = (t_0, t_f]$, $f : \Omega \times I \rightarrow \mathbb{R}^d$ and $u_0 = \Omega \times t_0 \rightarrow \mathbb{R}^d$, the Stokes problem for $u : \Omega \times I \rightarrow \mathbb{R}^d$ is

$$
\partial_t u + \nabla \cdot \sigma = f \quad \text{in } \Omega,
$$
$$
\nabla \cdot u = 0 \quad \text{in } \Omega,
$$
$$
u = 0 \quad \text{on } \Gamma = \partial \Omega,
$$
$$
\int_{\Omega} p \, dx = 0,
$$

where $\sigma = p\mathbb{I} - \nabla u$. 
Define the spaces,

\[ V_h = \{ v_h \in [L^2(\mathcal{T})]^d \, , \, v_h \in [P_k(K)]^d \quad \forall K \in \mathcal{T} \} \]
\[ \tilde{V}_h = \{ \tilde{v}_h \in [L^2(\mathcal{F})]^d \, , \, \tilde{v}_h \in [P_k(F)]^d \quad \forall F \in \mathcal{F} \} \]
\[ Q_h = \{ q_h \in L^2(\mathcal{T}) \, , \, q_h \in P_{k-1}(K) \quad \forall K \in \mathcal{T} \} \]
\[ \tilde{Q}_h = \{ \tilde{q}_h \in L^2(\mathcal{F}) \, , \, \tilde{q}_h \in P_k(F) \quad \forall F \in \mathcal{F} \} \]
Weak formulation

Find \((u, \bar{u}, p, \bar{p}) \in V_h \times \bar{V}_h \times Q_h \times \bar{Q}_h\) s.t.
\(\forall (v, \bar{v}, q, \bar{q}) \in V_h \times \bar{V}_h \times Q_h \times \bar{Q}_h,\)

\[
\sum_{K \in T} \int_K \nabla u : \nabla v \, dx + \sum_{K \in T} \int_{\partial K} (\bar{u} - u) \cdot \frac{\partial v}{\partial n} \, ds - \sum_{K \in T} \int_K p \nabla \cdot v \, dx \\
+ \sum_{K \in T} \int_{\partial K} \hat{\sigma} n \cdot (v - \bar{v}) \cdot ds = \sum_{K \in T} \int_K f \cdot v \, dx
\]

and

\[
\sum_{K \in T} \int_K u \cdot \nabla q \, dx + \sum_{K \in T} \int_{\partial K} \hat{u} \cdot n(\bar{q} - q) \, ds - \int_\Gamma \bar{u} \cdot n\bar{q} \, ds = 0.
\]
Numerical Fluxes

\[ \hat{\sigma} = -\nabla u + \bar{p} l - \frac{\alpha_v}{h_K} (\bar{u} - u) \otimes n, \]

\[ \hat{u} = u - \alpha_p h_K (\bar{p} - p) n. \]
\[ \hat{\sigma} = -\nabla u + \bar{p}l - \frac{\alpha_v}{h_K} (\bar{u} - u) \otimes n, \]
\[ \hat{u} = u - \alpha_p h_K (\bar{p} - p) n. \]

\[ V_h = \{ v_h \in [L^2(T)]^d, v_h \in [P_k(K)]^d \quad \forall K \in T \} \]

\[ Q_h = \{ q_h \in L^2(T), q_h \in P_{k-1}(K) \quad \forall K \in T \} \]

\( \alpha_p \) can be set to zero.
Some Insights to Weak Formulation

\[
\sum_{K \in \mathcal{T}} \int_K \nabla u : \nabla v \, dx + \sum_{K \in \mathcal{T}} \int_{\partial K} (\bar{u} - u) \cdot \frac{\partial v}{\partial n} \, ds - \sum_{K \in \mathcal{T}} \int_K p \nabla \cdot v \, dx \\
+ \sum_{K \in \mathcal{T}} \int_{\partial K} \hat{\sigma}_n \cdot (v - \bar{v}) \cdot ds = \sum_{K \in \mathcal{T}} \int_K f \cdot v \, dx
\]
Some Insights to Weak Formulation

\[
\sum_{K \in T} \int_K \nabla u : \nabla v \, dx + \sum_{K \in T} \int_{\partial K} (\bar{u} - u) \cdot \frac{\partial v}{\partial n} \, ds - \sum_{K \in T} \int_K p \nabla \cdot v \, dx \\
+ \sum_{K \in T} \int_{\partial K} \hat{\sigma} n \cdot (v - \bar{v}) \cdot ds = \sum_{K \in T} \int_K f \cdot v \, dx
\]

Setting \( \bar{v} = 0 \), momentum balance subject to b.c. provided by \( \bar{u} \)
Some Insights to Weak Formulation

\[ \sum_{K \in T} \int_K \nabla u : \nabla v \, dx + \sum_{K \in T} \int_{\partial K} (\bar{u} - u) \cdot \frac{\partial v}{\partial n} \, ds - \sum_{K \in T} \int_K p \nabla \cdot v \, dx + \sum_{K \in T} \int_{\partial K} \hat{\sigma} n \cdot (v - \bar{v}) \cdot ds = \sum_{K \in T} \int_K f \cdot v \, dx \]

Setting \( \bar{v} = 0 \), momentum balance subject to b.c. provided by \( \bar{u} \)

Setting \( v = 0 \), weak continuity of \( \hat{\sigma} \) across facets
Some Insights to Weak Formulation

\[
\sum_{K \in T} \int_K u \cdot \nabla q \, dx + \sum_{K \in T} \int_{\partial K} \hat{u} \cdot n(\bar{q} - q) \, ds - \int_{\Gamma} \bar{u} \cdot n\bar{q} \, ds = 0.
\]
Some Insights to Weak Formulation

\[
\sum_{K \in T} \int_K u \cdot \nabla q \, dx + \sum_{K \in T} \int_{\partial K} \hat{u} \cdot n (\bar{q} - q) \, ds - \int_{\Gamma} \bar{u} \cdot n \bar{q} \, ds = 0.
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Setting $\bar{q} = 0$, enforcing the continuity equation locally
Some Insights to Weak Formulation

\[ \sum_{K \in T} \int_{K} u \cdot \nabla q \, dx + \sum_{K \in T} \int_{\partial K} \hat{u} \cdot n(\bar{q} - q) \, ds - \int_{\Gamma} \bar{u} \cdot n\bar{q} \, ds = 0. \]

Setting \( \bar{q} = 0 \), enforcing the continuity equation locally

Setting \( q = 0 \), weak continuity of \( \hat{u} \) across facets
$H(\text{div})$-conforming: normal component of $u$ is continuous across inter-element boundaries
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Set $v, \bar{v}, q = 0$ and sum the weak formulation equations to see.
Properties

\( H(div) \)-conforming: normal component of \( u \) is continuous across inter-element boundaries

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Pointwise divergence-free: \( \nabla \cdot u = 0 \)
**Properties**

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Pointwise divergence-free: $\boldsymbol{\nabla} \cdot u = 0$

Since $q, \boldsymbol{\nabla} \cdot u \in P_{k-1}(K)$, it follows.
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Mass Conservation: \( [[u]] = 0 \) at interior faces and \( u \cdot n = \bar{u} \cdot n \) at boundary faces.
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Momentum Conservation: \( \frac{d}{dt} \int_K u \, dx = \int_K f \, dx - \int_{\partial K} \hat{\sigma} n \, ds \)

Global energy stability: \( \frac{d}{dt} \int_K |u|^2 \, dx \leq 0 \).