Partial Differential Equations

ODE : $u' = f(t, u(t))$ → one independent variable

PDE : $u = u(x, y, z, t, \dots)$ → more than one independent variables

Example

1) $u_{xx} + u_{yy} = 0$

Transport Eqn
homogeneous linear PDE

2) $u_t + u u_x + u_{xxx} = 0$

Dispersive wave eqn
homogeneous nonlinear PDE

3) $\cos(xy^2) u_x - y^2 u_y = \tan(x^2 + y^2)$

Inhomogeneous linear PDE

The order of PDE is the highest derivative that appears.

Consider $\mathcal{L} u = g$
↳ linear derivative operator

if $g = 0$ homogeneous linear PDE
$g \neq 0$ inhomogeneous linear PDE

Quasi-linear PDEs : Contain highest-order derivatives of u linearly

$u_{xx} + u_{yy} = |\nabla u|^2 u$ → quasilinear

$u_t + u u_x = 0 \rightarrow$ quasi linear

$u_x^2 + u_y^2 = 1 \rightarrow$ not quasi linear

Semi linear PDE: The term with the highest-order derivatives of u does not depend of u and its lower order derivatives

$u_t + \underline{u_{xxx}} + u u_x = 0 \rightarrow$ semilinear

$u_t + \underline{u u_x} = 0 \rightarrow$ quasi linear but not semilinear

A PDE is fully nonlinear if the highest-order derivatives of u appear nonlinearly in the equation.

$u_x^2 + u_y^2 = 1 \rightarrow$ fully nonlinear

$\nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0 \rightarrow$ fully nonlinear

Well-posed PDEs of proper initial and boundary conditions satisfies the following conditions:

- Existence
- Uniqueness
- Stability

Classification of Second Order PDEs

$$a u_{xx} + b u_{xy} + c u_{yy} + d u_x + e u_y + f u + g = 0$$

$\Delta = b^2 - 4ac$

+ $\Delta > 0 \rightarrow$ hyperbolic

Wave eqn: $u_{tt} = c^2 u_{xx} \quad -\infty < x < \infty \quad c \neq 0$

The form of the general solution:

$$u(x,t) = \underbrace{f(x+ct)}_{\text{travelling to left at speed } c} + \underbrace{g(x-ct)}_{\text{travelling to right at speed } c}$$

- time dependent
- describe conservative physical processes such as convection that are not evolving toward to steady state
- well-posed for $t>0$ & $t<0$
- Energy is constant

$\Delta = 0 \rightarrow$ parabolic

Heat Epn: $u_t = K u_{xx}$

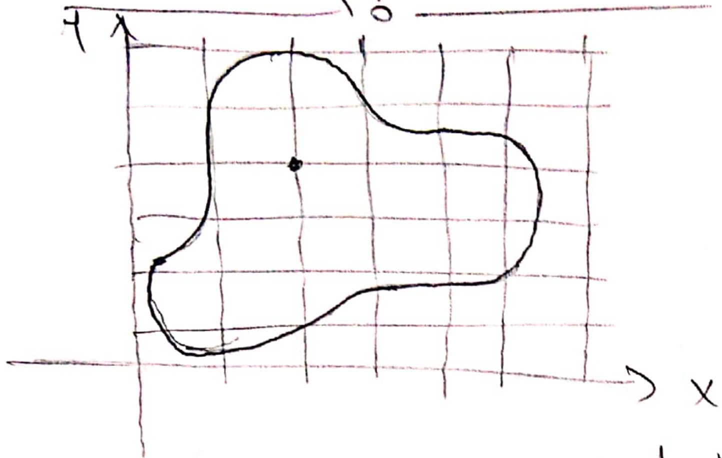
- describe time dependent dissipative physical process such as diffusion
- well-posed for $t>0$
- have maximum principle (max value of mitoal and domain boundary)

* $\Delta < 0 \rightarrow$ elliptic

Laplce Epn: $u_{xx} + u_{yy} = 0$

- Describe processes that have already reached steady state.

Finite Difference Method



- Replace the region over which the independent variables by a finite grid of points at which the dependent variable is approximated
- Partial Derivatives in PDE at each grid point are approximated from neighbouring values by using Taylor's Thm.
- Finally we seek numerical soln at mesh points

Taylor Series:

$$u(x_0+h) = u(x_0) + h\,u_x(x_0) + \frac{h^2}{2!}u_{xx}(x_0) + \frac{h^3}{3!}u_{xxx}(x_0)$$
$$+ \cdots + \frac{h^{n-1}}{(n-1)!}u_{(n-1)}(x_0) + O(h^n)$$

$x_0,\ x_0+h \rightarrow$ grid points

$u(x_0),\ u(x_0+h) \rightarrow$ known.

Truncate after the first derivative

(*) $$u(x_0+h) = u(x_0) + h\,u_x(x_0) + O(h^2)$$

$$\Rightarrow\quad u_x(x_0) \approx \frac{u(x_0+h)-u(x_0)}{h} \rightarrow$$ 1. order FD approximation (Forward)

Now, consider

(**) $u(x_0-h) = u(x_0) - h\,u'(x_0) + O(h^2)$

$$u'(x_0) \approx \frac{u(x_0) - u(x_0-h)}{h}$$ → 1. order F.D approximation (Backward)

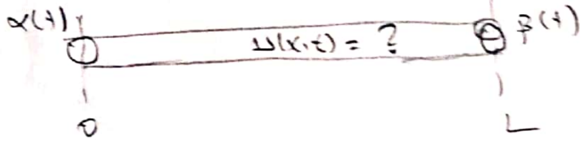
Substract (**) from (*)

$$u'(x_0) \approx \frac{u(x_0+h) - u(x_0-h)}{2h}$$ → 2. order FD approximation (Center Difference)

$$u''(x_0) \approx \frac{u(x_0+h) - 2u(x_0) + u(x_0-h)}{h^2}$$ → centered difference

Parabolic PDEs

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le L \quad \text{with}$$



Boundary conditions

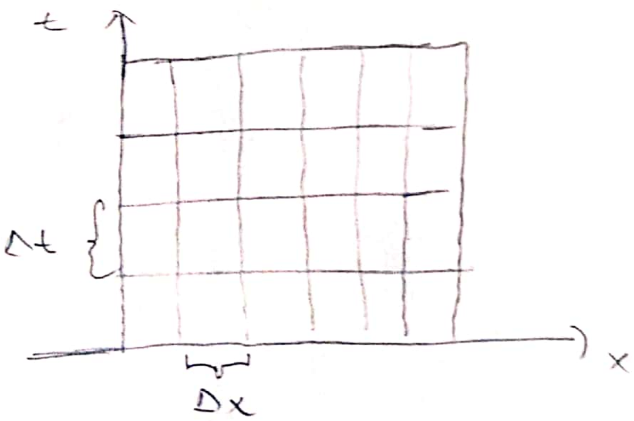
$u(0,t) = \alpha(t)$

$u(L,t) = \beta(t)$

Initial condition

$u(x,0) = u_0(x)$

α : thermal conductivity



$x_j = x_0 + j\Delta x$

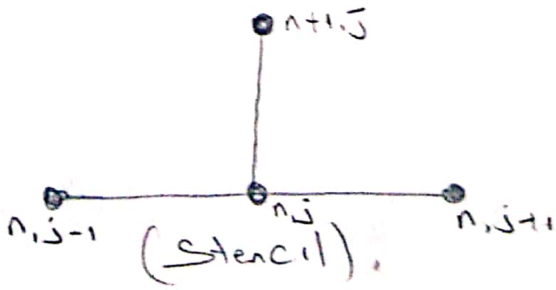
$t_n = t_0 + n\Delta t$

At mesh points

$u(x_j, t_n) = U_j^n \approx w_j^n$

Exact sol'n ← U_j^n; w_j^n → Approximate sol'n

Apply forward difference in time and centered difference in space



(Stencil).

$$\frac{w_j^{n+1} - w_j^{n}}{\Delta t} = \kappa \frac{w_{j+1}^{n} - 2 w_j^{n} + w_{j-1}^{n}}{\Delta x^2}$$

Set $r = \kappa \dfrac{\Delta t}{\Delta x^2}$

$$w_j^{n+1} = r\, w_{j+1}^{n} + (1-2r)\, w_j^{n} + r\, w_{j-1}^{n} \qquad \text{(Explicit)}$$

with $w_j^0 = u_0(x_j)$

$w_0^n = \alpha(t_n) \qquad w_{N+1}^n = \beta(t_n)$

Let

$$w^k = \begin{bmatrix} w_1^k \\ w_2^k \\ \vdots \\ w_N^k \end{bmatrix}$$

$$w^{k+1} = \begin{bmatrix} w_1^{k+1} \\ w_2^{k+1} \\ \vdots \\ w_N^{k+1} \end{bmatrix} = \overbrace{\begin{bmatrix} 1-2r & r & & & \\ r & 1-2r & r & & \\ & r & 1-2r & r & \\ & & \ddots & \ddots & r \\ & & & r & 1-2r \end{bmatrix}}^{A = I - rB} \begin{bmatrix} w_1^k \\ w_2^k \\ \vdots \\ w_N^k \end{bmatrix}$$

$$+ \underbrace{\begin{bmatrix} r\, w_0^k \\ \vdots \\ 0 \\ \vdots \\ r\, w_{N+1}^k \end{bmatrix}}_{b_k}$$

$$B = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}$$

Stability Analysis :

- By using the matrix A .

Consider $Ae_i = \lambda e_i$

- if $|\lambda_i| > 1$ $\quad \|A^k e_i\| = \|\lambda^k e_i\| \to \infty$
- if $|\lambda_i| \le 1$ $\quad \|A^k z\| \to 0$ for any vector z

$$A = \begin{bmatrix} a & b & & 0 \\ c & a & b & \\ & \ddots & \ddots & b \\ 0 & & c & a \end{bmatrix} \in \mathbb{R}^{n\times n} \Rightarrow \lambda_i = a + 2\sqrt{bc}\cos\frac{i\pi}{N+1}$$

$i = 1, 2, \ldots n$

For our matrix,

$$\lambda_i = (1-2r) + 2\sqrt{r^2}\cos\frac{i\pi}{N+1}$$

$$= 1 - 2r + 2r\cos\frac{i\pi}{N+1}$$

$$= 1 + 2r\Big(\underbrace{\cos\frac{i\pi}{N+1} - 1}_{1-2\sin^2\frac{i\pi}{N+1}}\Big) = 1 - 4r\sin^2\frac{i\pi}{2(N+1)}$$

$$-1 \le \lambda_i = 1 - 4r\sin^2\frac{i\pi}{2(N+1)} \le 1$$

$$0 \le 2r\sin^2\frac{i\pi}{2(N+1)} \le 1 \Rightarrow r \le \frac{1}{2\sin^2\frac{i\pi}{2(N+1)}} \le \frac{1}{2}$$

- Von-Neumann Stability .

Take the soln $u(x,t) = \eta(t)\, e^{i\omega x}$

Evaluate it at (x_j, t_n) $\quad u_j^n = \eta(t_n)\, e^{i\omega j\Delta x} = \eta^n e^{i\omega j \Delta x}$

Replace the discrete scheme.

For stability $|\eta| \le 1$

\# Consider $u_t = u_{xx}$

- Apply BD in time and centered difference in space

$$\frac{w_j^{n+1} - w_j^n}{\Delta t} = \frac{w_{j-1}^{n+1} - 2w_j^{n+1} + w_{j+1}^{n+1}}{\Delta x^2} \quad \text{(implicit)}$$

- Apply θ-method in time and centered difference in space

$$\frac{w_j^{n+1} - w_j^n}{\Delta t} = (1-\theta)\frac{w_{j-1}^n - 2w_j^n + w_{j+1}^n}{\Delta x^2} + \theta\frac{w_{j-1}^{n+1} - 2w_j^{n+1} + w_{j+1}^{n+1}}{\Delta x^2}$$

$\theta = 0 \to$ FTCS

$\theta = 1 \to$ BTCS

$\theta = \frac{1}{2} \to$ CN

\# In case $u_x(0,t) = g(t)$ (Neuman BC)

Apply FD $\quad u_x(0,t) \approx \dfrac{w_1^n - w_0^n}{\Delta x} = g(t_n)$

CD $\quad u_x(0,t) \approx \dfrac{w_1^n - w_{-1}^n}{2\Delta x} = g(t_n)$

Ex $\quad u_{xx} + u_{yy} = x + y \qquad 0 \le x \le 1$

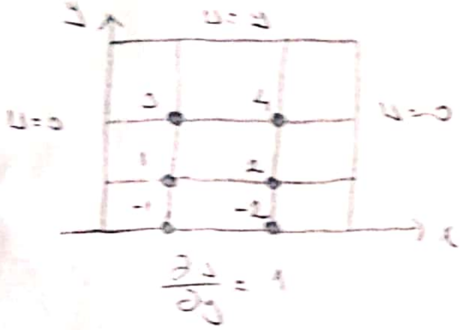
$$0 \le y \le 1$$

$\frac{\partial u}{\partial y} = 1$ on $y = 0$ and $u = y$ on $y = 1$

$u = 0$ on $x = 0$ and $x = 1$

Derive a second order fd approximations to differential eqn Use the mesh size $\Delta x = \Delta y = \frac{1}{3}$ and construct the linear system of eqns.

Soln



$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j-1} - 2u_{ij} + u_{i,j+1}}{h^2} = x_i + y_j$$

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij} = h^2(x_i + y_j)$$

For point 1 $\quad u_2 + 0 + u_3 + u_{-1} - 4u_1 = h^2\left(\frac{1}{3} + \frac{1}{3}\right)$

For point 2 $\quad 0 + u_1 + u_4 + u_{-2} - 4u_2 = h^2\left(\frac{2}{3} + \frac{1}{3}\right)$

For point 3 $\quad u_4 + 0 + 1 + u_1 - 4u_3 = h^2\left(\frac{1}{3} + \frac{2}{3}\right)$

For point 4 $\quad 0 + u_3 + 1 + u_2 - 4u_4 = h^2\left(\frac{2}{3} + \frac{2}{3}\right)$

Unknowns are $u_{-1}, u_{-2}, u_1, u_2, u_3, u_4 \Rightarrow 6$ unknown

Apply Foward difference for $\frac{\partial u}{\partial y} = 1$

For point -1 $\quad \frac{u_1 - u_{-1}}{h} = 1 \Rightarrow u_{-1} = u_1 - h$

For point -2 $\quad \frac{u_2 - u_{-2}}{h} = 1 \Rightarrow u_{-2} = u_2 - h$

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$h = 1/3$ and using $u_1 = u_1 - h$ & $u_{-2} = u_2 - h$

$$-3u_1 + u_2 + u_3 = 11/27$$

$$u_1 - 3u_2 + u_4 = 12/27$$

$$u_1 - 4u_3 + u_4 = -24/27$$

$$u_2 + u_3 - 4u_4 = -23/27$$

$$\begin{bmatrix} -3 & 1 & 1 & 0 \\ 1 & -3 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \frac{1}{27} \begin{bmatrix} 11 \\ 12 \\ -24 \\ -23 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 0 & -4 & 1 & 1 & 0 \\ 0 & 1 & 1 & -4 & 0 & 1 \\ 0 & 0 & 1 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 1 & -4 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{-1} \\ u_{-2} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \frac{1}{27} \begin{bmatrix} 2 \\ 3 \\ -24 \\ -23 \\ 9 \\ 9 \end{bmatrix}$$

$u_1 = -0.2012$ $\quad$ $u_4 = -0.260$

$u_2 = -0.3025$ $\quad$ $u_{-1} = -0.5345$

$u_3 = 0.1064$ $\quad$ $u_{-2} = -0.6358$

• Hyperbolic Eqns

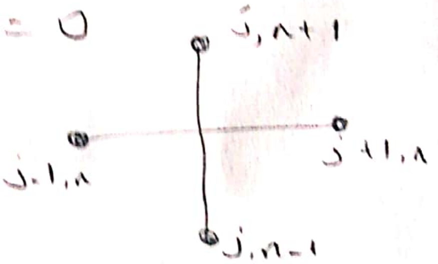
Ex: $\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial (u^2)}{\partial x} = 0$ Burger's Eqn

$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ → Wave Eqn

$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$ → Advection Eqn.

Apply FTBS to advection Eqn

$$\frac{w_j^{n+1} - w_j^n}{\Delta t} + a\frac{w_j^n - w_{j-1}^n}{\Delta x} = 0$$



Leapfrog Scheme (CTCS)

$$\frac{w_j^{n+1} - w_j^{n-1}}{2\Delta t} + a\frac{w_{j+1}^n - w_{j-1}^n}{2\Delta x} = 0$$

Elliptic Eqns

$\nabla \cdot (\nabla u) = \Delta u = u_{xx} + u_{yy} = f$ → Poisson Eqn.

$f = 0$ → Laplace Eqn.

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} = f(x_i, y_j)$$

5-point stencil