PhD Qualifying Exam

Numerical Optimisation
May, 2011

Name SURNAME: ID:

---

1. Let the unconstrained problem be $\underset{\vec{x}\in\mathbb{R}^2}{\text{minimise}}\, f(\vec{x}) = x_1^2 + 100x_2^2$ and $\vec{x}_0 = (1,1)^T$.

   (a) Starting with $\vec{x}_0$, apply the method of *steepest decsent* to find $\vec{x}_1$.

   (b) Starting with $\vec{x}_0$, apply the *Newton* method to find $\vec{x}_1$.

   (c) For both methods in (1a) and (1b), find the values of μ_1 and μ_2 (both being positive) such that

$$\begin{aligned} f(\vec{x} + \alpha\vec{p}) &\leq f(\vec{x}) + \mu_1 \alpha \nabla f(\vec{x})^T \vec{p} \\ \vec{p}^T \nabla f(\vec{x} + \alpha\vec{p}) &\geq \mu_2 \nabla f(\vec{x})^T \vec{p} \end{aligned}$$

   are satisfied, where α is the step length along the direction $\vec{p}$.
   In particular, find μ_1 and μ_2 if $\alpha = 1$ (unit step length) is acceptable in the line search.

PhD Qualifying Exam

Numerical Optimisation
May, 2011

Name SURNAME: ID:

---

2. Consider the linear programming in $\mathbb{R}^3$,

$$(\mathcal{LP}_3) \quad \left\{ \begin{array}{lrcl} \text{minimise} & 3x_1 + 3x_2 - x_3 & & \\ \text{subject to} & x_1 + 2x_2 & = & 6 \\ & -2x_1 - 2x_2 + x_3 & = & 0 \\ & x_2 & \leq & 4. \end{array} \right.$$

   (a) Find an equivalent *reduced* programming $(\mathcal{LP}_2)$ in $\mathbb{R}^2$, by removing x_3.

   (b) Write down the standard form $(\mathcal{LP}_{\text{st}})$ of the reduced problem $(\mathcal{LP}_2)$.

   (c) Solve $(\mathcal{LP}_{\text{st}})$ by using the *simplex* method using *tableaus*, and emphasise the optimal solution to the original linear programming $(\mathcal{LP}_3)$.

   (d) Find the dual $(\mathcal{LP}_2^*)$ of $(\mathcal{LP}_2)$.

   (e) Solve the dual problem $(\mathcal{LP}_2^*)$ and justify your answer in (2c).

PhD Qualifying Exam

Numerical Optimisation
May, 2011

Name SURNAME: ID:

---

3. Consider the problem

$$\begin{aligned} \text{minimise} \quad & x_1 + x_2 \\ \text{subject to} \quad & -x_1^2 + x_2 \geq 0 \\ & x_1 \geq 0 \end{aligned}$$

(a) Use a graphical representation (do not directly solve the problem) to find the *global* optimal solution. Justify your answer.

(b) Using the *log-barrier* reformulation, solve the problem.

PhD Qualifying Exam

Numerical Optimisation
May, 2011

Name SURNAME: ID:

---

4. Consider the constrained problem

$$(\mathcal{P}) \quad \begin{cases} \text{minimise} & x_1^2 + x_2^2 \\ \text{subject to} & -x_1 - x_2 + 4 \leq 0 \qquad (*) \\ & x_1, x_2 \geq 0 \end{cases}$$

and let $\vec{x}_0 = (6, 0)^T$.

(a) Using an *active-set strategy* find and optimal solution to $(\mathcal{P})$. You may use the *Newton* or the *reduced Newton* direction, whichever is applicable.

(b) Is the solution you found a *unique* solution to $(\mathcal{P})$? Justify your answer.

(c) Using the *Lagrangian duality* by *relaxing* the first constraint $(*)$, find an optimal solution to $(\mathcal{P})$. Does the *strong duality* hold?

PhD Qualifying Exam

Numerical Optimisation
May, 2011

Name SURNAME: ID:

---

5. Consider the problem

$$(\mathcal{P}) \quad \begin{cases} \underset{x\in\mathbb{R}^n}{\text{minimise}} & f(x) \\ \text{subject to} & g(x) \geq 0 \end{cases} .$$

   (a) Write down the first-order necessary optimality conditions for $(\mathcal{P})$.

   (b) Consider the *trust-region* subproblem

$$(\mathcal{TR}) \quad \begin{cases} \underset{x\in\mathbb{R}^n}{\text{minimise}} & c^T x + \frac{1}{2}x^T B x \\ \text{subject to} & \|x\|_2 \leq \Delta \end{cases} .$$

   for some scalar $\Delta > 0$. By noting that the constraint is equivalent to $\frac{1}{2}x^T x \leq \frac{1}{2}\Delta^2$ and using your asnwer in part (5a), show that the solution x^* to this trust-region problem $(\mathcal{TR})$ necessarily satisfies

$$(B + \lambda^* I)x^* = -c$$

   where I is the identity matrix and the scalar $\lambda^* \geq 0$. Show, in addition, that either $\lambda^* = 0$ or $\|x\|_2 = \Delta$.

   (c) Find the solution to the trust-region subproblem $(\mathcal{TR})$ with data

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \Delta = \frac{5}{12}.$$

   Hint: $\lambda = 2$ is a root of $\frac{1}{(1+\lambda)^2} + \frac{1}{(2+\lambda)^2} = \frac{25}{144}$.

PhD Qualifying Exam

Numerical Optimisation
May, 2011

Name SURNAME: ID:

---

6. In the following we distinguish between *barrier* methods and *penalty* methods.

   (a) Among these two classes (barrier and penalty), which one can be refered to as *Interior Point Methods*, which one as *Exterior Point Methods*? Give reasoning for your answer.

   (b) Give an example for a *barrier* function, and an example for a *penalty* function.

   (c) State, by a *pseudo-code* (or a *flow chart*), how *interior point methods* basically work.

   (d) What kind of technical (numerical) refinements dou you know for *interior point methods*? Briefly explain.

   (e) In what context and for what reasons do we need *Merit function*?