Monotonicity and additivity properties of interval solutions for convex interval games

Rodica Branzei
Faculty of Computer Science, Alexandru Ioan Cuza University, Iaşi, Romania

based on

- **Alparslan Gök S.Z., Branzei R. and Tijs S., Convex interval games, Tilburg University, CentER DP 37 (2008)**
- **Yanovskaya E., Branzei R. and Tijs S., Monotonicity properties of interval solutions and the Dutta-Ray solution for convex interval games, Tilburg University, CentER DP 102 (2008)**

Outline

Interval cooperative games versus traditional cooperative games

- $N = \{1, 2, ..., n\}$ - the set of players
- $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$ v is the characteristic function of the game.

  G^N - the family of coalitional (TU) games with player set N
- $w : 2^N \to I(\mathbb{R})$, $w(\emptyset) = [0, 0]$

  IG^N - the family of all interval games with player set N

Example (LLR-game):

Let $< N, w >$ be an interval game with $w(1, 3) = w(2, 3) = w(1, 2, 3) = J \succcurlyeq [0, 0]$ and $w(S) = [0, 0]$ otherwise.

Interval Calculus

$I(\mathbb{R})$ - the set of all closed intervals in $\mathbb{R}$
$I(\mathbb{R})^N$ - the set of all n-dimensional vectors with elements in $I(\mathbb{R})$
Let $I, J \in I(\mathbb{R})$ with $I = [\underline{I}, \overline{I}]$, $J = [\underline{J}, \overline{J}]$, $|I| = \overline{I} - \underline{I}$ and $\alpha \in \mathbb{R}_+$.
Then,

- $I + J = [\underline{I} + \underline{J}, \overline{I} + \overline{J}]$
- $\alpha I = [\alpha \underline{I}, \alpha \overline{I}]$
- $I - J = [\underline{I} - \underline{J}, \overline{I} - \overline{J}]$ (defined only if $|I| \geq |J|$)
  (Moore (1979): $I - J = [\underline{I} - \overline{J}, \overline{I} - \underline{J}]$)
- $I \succcurlyeq J$, if and only if $\underline{I} \geq \underline{J}$ and $\overline{I} \geq \overline{J}$
- $I \preccurlyeq J$, if and only if $\underline{I} \leq \underline{J}$ and $\overline{I} \leq \overline{J}$

Convex interval games versus traditional convex games

A game $< N, v >$ is called convex (or supermodular) (Shapley (1971)) if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all $S, T \subset N$.

CG^N - the class of convex games with player set N.

We call a game $w \in IG^N$

- supermodular if

$$w(S) + w(T) \preccurlyeq w(S \cup T) + w(S \cap T) \text{ for all } S, T \in 2^N; \quad (a)$$

- convex if $< N, w >$ is supermodular and its length game $< N, |w| >$ is also supermodular.

CIG^N - the class of convex interval games with player set N.

Properties of solutions for traditional (convex) games

The core (Gillies (1959)) $C(v)$ of $v \in G^N$ is defined by

$$C(v) = \left\{ x \in \mathbb{R}^N | \sum_{i \in N} x_i = v(N); \sum_{i \in S} x_i \geq v(S) \text{ for each } S \in 2^N \right\}.$$

$C(v) \neq \emptyset$ for each $v \in CG^N$.

$C : G^N \twoheadrightarrow \mathbb{R}^N$ is superadditive.

$C : CG^N \twoheadrightarrow \mathbb{R}^N$ is additive.

Properties of solutions for traditional (convex) games

$\Pi(N)$: set of permutations $\sigma : N \to N$
$P_\sigma(i) = \{r \in N | \sigma^{-1}(r) < \sigma^{-1}(i)\}$: set of predecessors of i in σ

The marginal vector $m^\sigma(v)$ of v gives player i:
$m_i^\sigma(v) = v(P_\sigma(i) \cup \{i\}) - v(P_\sigma(i))$

The Weber set (Weber (1988)) $W(v)$ of a game $v \in G^N$ is the convex hull of the $n!$ marginal vectors $m^\sigma(v)$, corresponding to $n!$ permutations $\sigma \in \Pi(N)$.

$C(v) \subset W(v)$ for each $v \in G^N$.

$C(v) = W(v)$ for each $v \in CG^N$.

Properties of solutions for traditional (convex) games

The Shapley value (Shapley (1953)) $\phi(v)$ of a game $v \in G^N$ is the average of the marginal vectors of the game, i.e.,

$$\phi(v) := \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^{\sigma}(v).$$

$\phi : G^N \to \mathbb{R}^N$ is additive.

$\phi(v) \in C(v)$ for each $v \in CG^N$.

Monotonicity properties of TU game values

Convex monotonicity (CvM). If $\langle N, v\rangle$, $\langle N, v'\rangle$, $\langle N, v'-v\rangle \in CG^N$, and $v'(S) \geq v(S)$ for all $S \subset N$, then $\varphi(N, v') \geq \varphi(N, v)$.

Aggregate monotonicity If $v'(N) > v(N)$ and $v'(S) = v(S)$ for all $S \subsetneq N$, then $\varphi(N, v') \geq \varphi(N, v)$.

Coalitional monotonicity. For each coalition $S \subset N$, $v'(S) > v(S)$ and $v'(T) = v(T)$ for all $T \neq S$ imply $\varphi_i(N, v') \geq \varphi_i(N, v)$ for all $i \in S$.

Monotonicity properties of TU game values

Contribution monotonicity (CM). For each $i \in N$ inequalities $v'(S \cup \{i\}) - v'(S) \geq v(S \cup \{i\}) - v(S)$ for all $S \not\ni i$ imply $\varphi_i(N, v') \geq \varphi_i(N, v)$.

Weak contribution monotonicity (WCM) (Hokari, van Gellekom (2002)) If for all $i \in N$ and all coalitions $S \not\ni i$ the inequalities $v'(S \cup \{i\}) - v'(S) \geq v(S \cup \{i\}) - v(S)$ hold, then $\varphi(N, v') \geq \varphi(N, v)$.

Note that all these properties were defined for games with the same sets of players. It is clear that

$$CM \implies WCM \implies AM. \tag{1}$$

Let us check where convex monotonicity is placed in relations (1).

Proposition 1: On the class of convex games GC^N

$$WCM \implies CvM \implies AM.$$

Proof. Let $\langle N, v\rangle, \langle N, v'\rangle, \langle N, v'-v\rangle$ be convex games such that $v'(S) \geq v(S)$ for all $S \subset N$. Then for all $i \in N$ and $S \not\ni i$

$$v'(S \cup \{i\}) - v'(S) \geq v(S \cup \{i\}) - v(S). \tag{2}$$

If a value φ on CG^N satisfies weak contribution monotonicity, then $\varphi(N, v') \geq \varphi(N, v)$, and φ satisfies convex monotonicity.
Let now φ be any value on the class CG^N that satisfies convex monotonicity. Then for games $\langle N, v\rangle, \langle N, v'\rangle$ inequalities (2) hold, inclusively for those such that $v(S) = v'(S)$ for all $S \subsetneq N$, $v'(N) > v(N)$, implying $\varphi(N, v') \geq \varphi(N, v)$. □

The (constrained) egalitarian solution $DR(v)$ of $v \in G^N$ for TU games was defined (Dutta and Ray (1989)) as the unique Lorenz maximal allocation in the Lorenz core. $DR(v)$ can be empty, but it exists if $v \in CG^N$. For each convex game $\langle N, v \rangle$ the Dutta–Ray solution is the unique allocation in the core which Lorenz dominates all other core allocations. The Dutta–Ray solution on the class of convex TU games possesses many attractive properties. In particular, Hokari and van Gellekom (2002) proved that the DR solution over the class of convex games satisfies *weak contribution monotonicity*.

Monotonicity properties of TU game values

The last monotonicity property compares players' payoffs with respect to solution vectors in the initial game and its subgames:

Population monotonicity. If $\langle N, v\rangle$ is a convex game and $N' \subset N$, then $\varphi_i(N, v) \geq \varphi_i(N', v)$ for all $i \in N'$, where $\langle N', v\rangle$ is the subgame of $\langle N, v\rangle$.

This property assures the existence of population monotonic allocation schemes, (Sprumont (1990)). Recall that for a game $v \in G^N$ a scheme $a = (a_{iS})_{i\in S, S\in 2^N\setminus\{\emptyset\}}$ of real numbers is a population monotonic allocation scheme of v if

(i) $\sum_{i\in S} a_{iS} = v(S)$ for all $S \in 2^N \setminus \{\emptyset\}$,

(ii) $a_{iS} \leq a_{iT}$ for all $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subset T$ and for each $i \in S$.

We notice that convexity of v is a sufficient condition for the existence of population monotonic allocation schemes.

Classes of cooperative interval games

We call a game $w \in IG^N$

- size monotonic if $< N, |w| >$ is monotonic, i.e. $|w|(S) \leq |w|(T)$ for all $S, T \in 2^N$ with $S \subset T$.
- $\mathcal{I}$-balanced if $\mathcal{C}(w) \neq \emptyset$.

$SMIG^N$ - the class of size monotonic games with player set N.

$\mathcal{I}BIG^N$: class of interval balanced games with player set N

$CIG^N \subset (SMIG^N \cap \mathcal{I}BIG^N)$

Properties of solutions for (convex) interval games

The interval core:

$$\mathcal{C}(w) = \left\{ (I_1, \ldots, I_n) \in I(\mathbb{R})^N \Big| \sum_{i \in N} I_i = w(N), \sum_{i \in S} I_i \succcurlyeq w(S),\ \forall S \subset N \right\}$$

payoff interval of player i: I_i
interval-payoff vector: $I = (I_1, I_2, \ldots, I_n)$
$\sum_{i \in N} I_i = w(N)$: efficiency condition
$\sum_{i \in S} I_i \succcurlyeq w(S)$: stability condition

$\mathcal{C} : IG^N \twoheadrightarrow I(\mathbb{R})^N$ is a superadditive map.

$\mathcal{C} : CIG^N \twoheadrightarrow I(\mathbb{R})^N$ is an additive map.

Properties of solutions for (convex) interval games

Let $w \in SMIG^N$.

- The interval marginal vector $m^\sigma(w)$ of w gives player i: $m_i^\sigma(w) = w(P_\sigma(i) \cup \{i\}) - w(P_\sigma(i))$
- The interval Weber set $\mathcal{W}$ is defined by $\mathcal{W}(w) = conv\,\{m^\sigma(w) | \sigma \in \Pi(N)\}$.
- The interval Shapley value $\Phi : SMIG^N \to I(\mathbb{R})^N$: $\Phi(w) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(w), \text{ for each } w \in SMIG^N \quad (b)$

$\mathcal{W}(w) \subset \mathcal{C}(w)$ for each $w \in CIG^N$.

$\Phi(w) \in \mathcal{C}(w)$ for each $w \in CIG^N$.

Proposition: The interval Shapley value $\Phi : SMIG^N \to I(\mathbb{R})^N$ is additive.

Proof: First, we show that for each $\sigma \in \Pi(N)$ the interval marginal operator $m^\sigma : SMIG^N \to I(\mathbb{R})^N$ is additive, i.e., for all $w_1, w_2 \in SMIG^N$, $m^\sigma(w_1 + w_2) = m^\sigma(w_1) + m^\sigma(w_2)$.
Let $\sigma \in \Pi(N)$ and $k \in N$. Then,

$$\begin{aligned}
m^\sigma_{\sigma(k)}(w_1 + w_2) &= (w_1 + w_2)(\sigma(1), \dots, \sigma(k)) \\
&- (w_1 + w_2)(\sigma(1), \dots, \sigma(k-1)) \\
&= w_1(\sigma(1), \dots, \sigma(k)) - w_1(\sigma(1), \dots, \sigma(k-1)) \\
&+ w_2(\sigma(1), \dots, \sigma(k)) - w_2(\sigma(1), \dots, \sigma(k-1)) \\
&= m^\sigma_{\sigma(k)}(w_1) + m^\sigma_{\sigma(k)}(w_2).
\end{aligned}$$

Now, using the additivity property of interval marginal operators we obtain that $\Phi : SMIG^N \to I(\mathbb{R})^N$ is an *additive* map.

Proposition: Let $w \in SMIG^N$ and let $\sigma \in \Pi(N)$. Then, $m_i^\sigma(w) = [m_i^\sigma(\underline{w}), m_i^\sigma(\overline{w})]$ for all $i \in N$.

Proposition: Let $w \in SMIG^N$ and let $\sigma \in \Pi(N)$. Then, $\Phi_i(w) = [\phi_i(\underline{w}), \phi_i(\overline{w})]$ for all $i \in N$.

Proof. From (b) and above Proposition we have

$$\Phi_i(w) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m_i^\sigma(w) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} [m_i^\sigma(\underline{w}), m_i^\sigma(\overline{w})] \quad =$$

$$\left[\frac{1}{n!} \sum_{\sigma \in \Pi(N)} m_i^\sigma(\underline{w}), \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m_i^\sigma(\overline{w}) \right] = [\phi_i(\underline{w}), \phi_i(\overline{w})].$$

Proposition: The interval Shapley value has the population monotonicity property on the class of convex interval games.
Proof. Let $w \in CIG^N$. We have to prove that for all $S, T \in 2^N$ such that $S \subset T$ and for each $i \in N$ the relation $\Phi_i(S, w_S) \preccurlyeq \Phi_i(T, w_T)$ holds, where (S, w_S) and (T, w_T) are the corresponding subgames. We know that $\Phi_i(w) = [\phi_i(\underline{w}), \phi_i(\overline{w})]$ for each $w \in CIG^N$ and for all $i \in N$. Further, the fact that the classical Shapley value ϕ has the population monotonicity property on CG^N implies that for each $S, T \in 2^N$ such that $S \subset T$ and for each $i \in N$, $\phi_i(S, \underline{w}_S) \leq \phi_i(T, \underline{w}_T)$ and $\phi_i(S, \overline{w}_S) \leq \phi_i(T, \overline{w}_T)$, from which follows

$$[\phi_i(S, \underline{w}_S), \phi_i(S, \overline{w}_S)] = \Phi_i(S, w_S) \preccurlyeq \Phi_i(T, w_T)$$

$$= [\phi_i(T, \underline{w}_T), \phi_i(T, \overline{w}_T)].$$

Interval population monotonic allocation schemes (pmas)

We say that for a game $w \in T\mathcal{I}BIG^N$ a scheme $A = (A_{iS})_{i \in S, S \in 2^N \setminus \{\emptyset\}}$ with $A_{iS} \in I(\mathbb{R})^N$ is a pmas of w if

(i) $\sum_{i \in S} A_{iS} = w(S)$ for all $S \in 2^N \setminus \{\emptyset\}$,

(ii) $A_{iS} \preccurlyeq A_{iT}$ for all $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subset T$ and for each $i \in S$.

We notice that convexity of w is a sufficient condition for the existence of interval pmas.

Interval population monotonic allocation schemes (pmas)

Example: Let $w \in CIG^N$ with $w(\emptyset) = [0, 0]$, $w(1) = w(2) = w(3) = [0, 0]$, $w(1, 2) = w(1, 3) = w(2, 3) = [2, 4]$ and $w(1, 2, 3) = [9, 15]$. It is easy to check that the interval Shapley value generates for this game the pmas depicted as

$$\begin{array}{c} \\ N \\ \{1,2\} \\ \{1,3\} \\ \{2,3\} \\ \{1\} \\ \{2\} \\ \{3\} \end{array} \left[\begin{array}{ccc} 1 & 2 & 3 \\ [3,5] & [3,5] & [3,5] \\ [1,2] & [1,2] & * \\ [1,2] & * & [1,2] \\ * & [1,2] & [1,2] \\ [0,0] & * & * \\ * & [0,0] & * \\ * & * & [0,0] \end{array} \right].$$

Square solutions and related results

Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ with $a \le b$.
Then, we denote by $a \square b$ the vector $([a_1, b_1], \ldots, [a_n, b_n]) \in I(\mathbb{R})^N$ generated by the pair $(a, b) \in \mathbb{R}^N$.
Let $A, B \subset \mathbb{R}^N$. Then, we denote by $A \square B$ the subset of $I(\mathbb{R})^N$ defined by $A \square B = \{a \square b | a \in A, b \in B, a \le b\}$.

For each $w \in IG^N$:

- $\mathcal{W}^{\square}(w) = W(\underline{w})\square W(\overline{w})$;
  $\mathcal{C}(w) \subset \mathcal{W}^{\square}(w)$.
- $\mathcal{C}^{\square}(w) = C(\underline{w})\square C(\overline{w})$ for each $w \in \mathcal{I}BIG^N$.
  $\mathcal{C}(w) = \mathcal{C}^{\square}(w)$ for each $w \in \mathcal{I}BIG^N$.
- Let $w \in SMIG^N$ and let $\sigma \in \Pi(N)$. Then,
  $m_i^{\sigma}(w) = [m_i^{\sigma}(\underline{w}), m_i^{\sigma}(\overline{w})]$ for all $i \in N$.

For each $w \in CIG^N$:

$\mathcal{C}(w) = \mathcal{W}^{\square}(w)$.

$\mathcal{W}(w) \subset \mathcal{W}^{\square}(w)$.

Proposition: $\mathcal{C} : CIG^N \twoheadrightarrow I(\mathbb{R})^N$ is an additive map.

Proof: The interval core is a superadditive map on IG^N.
Therefore, we need to show the subadditivity of the interval core, i.e. $\mathcal{C}(w_1 + w_2) \subset \mathcal{C}(w_1) + \mathcal{C}(w_2)$ for all $w_1, w_2 \in CIG^N$.
Note that $m^\sigma(w_1 + w_2) = m^\sigma(w_1) + m^\sigma(w_2)$ for each $w_1, w_2 \in CIG^N$. By definition of the square interval Weber set we have $\mathcal{W}^\square(w_1 + w_2) = W(\underline{w}_1 + \underline{w}_2)\square W(\overline{w}_1 + \overline{w}_2)$, implying

$$\mathcal{C}(w_1+w_2) = \mathcal{W}^\square(w_1+w_2) \subset \mathcal{W}^\square(w_1)+\mathcal{W}^\square(w_2) = \mathcal{C}(w_1)+\mathcal{C}(w_2).$$

Inheritence of monotonicity properties by interval values

An *interval game* is a triple $\langle N, (\underline{w}, \overline{w})\rangle$ where N is a finite set of players, $\underline{w}, \overline{w} : 2^N \to \mathbb{R}$ are a *lower* and a *upper* characteristic functions, respectively, such that for each coalition $S \subset N$, $\underline{w}(S) \leq \overline{w}(S)$. The TU games $\langle N, \underline{w}\rangle, \langle N, \overline{w}\rangle$ are called the *lower* and the *upper* games of the interval game $\langle N, (\underline{w}, \overline{w})\rangle$, respectively.

Let G^N be an arbitrary class of TU games with the player set N. Further we denote by IG^N the class of interval games with the player set N such that for any $\langle N, (\underline{w}, \overline{w})\rangle \in IG^N$ both the lower and upper games $\langle N, \underline{w}\rangle, \langle N, \overline{w}\rangle$ belong to the class G^N.

Denote by $X(N, \underline{w}), X(N, \overline{w})$ the sets of feasible payoff vectors of the lower and upper games, and by $Y(N, \underline{w}), Y(N, \overline{w})$ the sets of *efficient* payoff vectors, respectively:

$$X(N, \underline{w}) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i \leq \underline{w}(N)\},$$
$$X(N, \overline{w}) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i \leq \overline{w}(N)\},$$

$$Y(N, \underline{w}) = \{x \in X(N, \underline{w}) \mid \sum_{i \in N} x_i = \underline{w}(N)\},$$
$$Y(N, \overline{w}) = \{x \in X(N, \overline{w}) \mid \sum_{i \in N} x_i = \overline{w}(N)\},$$

Definition 1: A *single-valued solution (value)* ϕ for a class IG^N of interval games is a mapping assigning to each interval game $\langle N, (\underline{w}, \overline{w}) \rangle \in IG^N$ a pair of vectors $\phi(N, (\underline{w}, \overline{w})) = (x, y) \in \mathbb{R}^{2n}$ such that $x \in X(N, \underline{w}), y \in X(N, \overline{w})$ and $x \leq y$.

Definition 2: An interval value ϕ on a class of interval games IG^N is *generated by a TU game value* φ if

$$\phi(N,(\underline{w},\overline{w})) = (\varphi(N,\underline{w}),\varphi(N,\overline{w})). \tag{3}$$

Equality (3) implies that the inequality

$$\varphi(N,\underline{w}) \leq \varphi(N,\overline{w}) \tag{4}$$

should hold, and, hence, not all TU game values can be extended to the generated interval values, and even if a value can be extended, then only for some special classes of TU and interval games. In the sequel we consider only interval values generated by some known TU game values.

Consider the class CG^N of convex TU games with a finite set of players N. Define the class IG^N of *convex interval games* with the universal set of players N by the following way:

$$\langle N, (\underline{w}, \overline{w})\rangle \in CIG^N \iff \langle N, \overline{w}\rangle, \langle N, \underline{w}\rangle, \langle N, \overline{w} - \underline{w}\rangle \in CG^N$$

$$\text{and } \underline{w}(S) \leq \overline{w}(S) \text{ for all } S \subset N.$$

Given a vector $x \in \mathbb{R}^N$ and a coalition $S \subset N$, by x_S we denote the projection of the vector x on the subspace $\mathbb{R}^S$, and by $x(S)$ the sum $x(S) = \sum_{i \in S} x_i$.
An interval $[a_1, a_2]$ *dominates* an interval $[b_1, b_2]$, denoted by $[a_1, a_2] \succcurlyeq [b_1, b_2]$, if $a_1 \geq b_1, a_2 \geq b_2$. An interval vector $\mathbf{a} = ([a_1, a_1'], \ldots, [a_n, a_n'])$ *dominates* an interval vector $\mathbf{b} = ([b_1, b_1'], \ldots, [b_n, b_n'])$, $\mathbf{a} \succcurlyeq \mathbf{b}$, if $[a_i, a_i'] \succcurlyeq [b_i, b_i']$ for $i = 1, \ldots, n$.

Next we show which TU game values for convex games can be extended to the generated interval values and which ones can not. By $C(N, v)$ we denote the core of $\langle N, v\rangle$, and by $\mathcal{C}(N, w)$ the *interval core* (of the interval game $\langle N, (\underline{w}, \overline{w})\rangle$, $w = (\underline{w}, \overline{w})$):

$$\mathcal{C}(N, w) = \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N \,|\, x \in C(N, \underline{w}), y \in C(N, \overline{w}), x \leq y\}.$$

Given a TU value φ for the class CG^N, the existence of the generated by it interval value ϕ on CIG^N, i.e. the fulfilment of inequality (4) is equivalent to convex monotonicity property (CvM) of φ.

Relations (3) and Proposition 1 permit to check for what TU game values for convex games the generated interval values exist or not. It is well-known that the Shapley value satisfies contribution monotonicity. Therefore, there exists the interval Shapley value on the class of convex interval games (Alparslan Gök, Branzei and Tijs (2008)). By Proposition 1 the Dutta-Ray solution for classical convex games satisfies convex monotonicity providing the existence of the generated Dutta– Ray interval solution on the class of convex interval games. On the other hand, it is known that the prenucleolus and the τ-value on the class of convex games do not satisfy aggregate monotonicity (Hokari (2000), Hokari and van Gellekom (2002)). Therefore, the interval prenucleolus and the interval τ-value do not exist on the class CIG^N.

For interval values we demand that the properties hold both for lower and upper games. Let ϕ be an interval value for the class CIG^N of interval convex games. The following definitions are the extensions to interval convex games of the given above monotonicity properties of TU game values.

Aggregate monotonicity If $\langle N, (\underline{w}, \overline{w})\rangle$ and $\langle N, (\underline{w'}, \overline{w'})\rangle$ are interval convex games such that $\underline{w}(S) = \underline{w'}(S), \overline{w}(S) = \overline{w'}(S)$ for all $S \subsetneq N$, and $\underline{w'}(N) > \underline{w}(N), \overline{w'}(N) > \overline{w}(N)$, then $\phi(N, (\underline{w'}, \overline{w'})) \succcurlyeq \phi(N, (\underline{w}, \overline{w}))$.

Coalitional monotonicity. If for interval convex games $\langle N, (\underline{w}, \overline{w})\rangle$ and $\langle N, (\underline{w'}, \overline{w'})\rangle$ for some coalition $S \subset N$ the following inequalities hold: $\underline{w'}(S) > \underline{w}(S), \overline{w'}(S) > \overline{w}(S)$ and $\underline{w'}(T) = \underline{w}(T), \overline{w'}(T) = \overline{w}(T)$ for all $T \neq S$, then $\phi_i(N, (\underline{w'}, \overline{w'})) \succcurlyeq \phi_i(N, (\underline{w}, \overline{w}))$ for all $i \in S$.

Contribution monotonicity (CM). For interval convex games $\langle N, (\underline{w}, \overline{w})\rangle$ and $\langle N, (\underline{w'}, \overline{w'})\rangle$ and for each $i \in N$ inequalities $\underline{w'}(S \cup \{i\}) - \underline{w'}(S) \geq \underline{w}(S \cup \{i\}) - \underline{w}(S)$, $\overline{w'}(S \cup \{i\}) - \overline{w}(S) \geq \overline{w}(S \cup \{i\}) - \overline{w}(S)$ for all $S \not\ni i$ imply $\phi_i(N, (\underline{w'}, \overline{w'})) \succcurlyeq \phi_i(N, (\underline{w}, \overline{w}))$.

Weak contribution monotonicity If for interval convex games $\langle N, (\underline{w}, \overline{w})\rangle$ and $\langle N, (\underline{w'}, \overline{w'})\rangle$, for all $i \in N$, and all coalitions $S \not\ni i$ the inequalities $\underline{w'}(S \cup \{i\}) - \underline{w'}(S) \geq \underline{w}(S \cup \{i\}) - \underline{w}(S)$, $\overline{w'}(S \cup \{i\}) - \overline{w'}(S) \geq \overline{w}(S \cup \{i\}) - \overline{w}(S)$ hold, then $\phi(N, (\underline{w'}, \overline{w'})) \succcurlyeq \phi(N, (\underline{w}, \overline{w}))$.

Population monotonicity. If $\langle N, (\underline{w}, \overline{w})\rangle$ is an interval convex game and $N' \subset N$, then $\phi_i(N, (\underline{w}, \overline{w})) \succcurlyeq \phi_i(N', (\underline{w}, \overline{w}))$ for all $i \in N'$, where $\langle N', (\underline{w}, \overline{w})\rangle$ is the subgame of $\langle N, (\underline{w}, \overline{w})\rangle$.

From the definitions it follows that all these properties are inherited by interval values generated by TU game values: if a value φ on the class of TU convex games CG^N satisfies one of the monotonicity properties, then the generated interval value ϕ on the class CIG^N satisfies the same interval property.
In particular, since the Shapley value and the Dutta–Ray solution on the class of convex games are population monotonic, we obtain that the interval Shapley value and the interval Dutta–Ray solution are population monotonic on the class of interval convex games as well.
This last monotonicity property provides the existence of interval population monotonic allocation schemes (Alparslan Gök, Branzei and Tijs (2008)).

References

Alparslan Gök, S.Z., Cooperative Interval Games, PhD Thesis, *Middle East Tehnical University, Turkey* (2009).

Dutta B. and Ray D., A concept of egalitarianism under participation constraints, *Econometrica* **57** (1989) 615-630.

Gillies D. B., Solutions to general non-zero-sum games. In: Tucker, A.W. and Luce, R.D. (Eds.), *Contributions to the theory of games IV, Annals of Mathematical Studies* **40**. *Princeton University Press, Princeton* (1959) pp. 47-85.

Hokari T., The nucleolus is not aggregate monotonic on the domain of convex games, *International Journal of Game Theory* **29** (2000) 133-137.

Hokari T. and van Gellekom A., Population monotonicity and consistency in convex games: Some logical relations, *International Journal of Game Theory* **31** (2002) 593-607.

References

Moore R., Methods and Applications of Interval Analysis, *SIAM Studies in Applied Mathematics, Philadelphia* (1979).

Shapley L.S., A value for *n*-person games, *Annals of Mathematics Studies* **28** (1953) 307-317.

Shapley L.S., Cores of convex games, *International Journal of Game Theory* **1** (1971) 11-26.

Sprumont Y., Population Monotonic Allocation Schemes for Cooperative Games with Transferable Utility, *Games and Economic Behavior* **2** (1990) 378-394.

Weber R., Probabilistic values for games, in Roth A.E. (Ed.), The Shapley Value: Essays in Honour of Lloyd S. Shapley, *Cambridge University Press, Cambridge* (1988) 101-119.