NUMERICAL METHODS FOR STOCHASTIC DIFFERENTIAL EQUATIONS

Part I: A brief introduction

Part II: Classical methods and results

Part III: Optimal L_2-approximation

Part IV: Optimal approximation at a single point

Part V: Outlook

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I. A brief introduction

Problem:
Approximation of the solution X of a (scalar) sde

$$(1) \quad dX(t) = a(t, X(t))\,dt + \sigma(t, X(t))\,dW(t), \quad t \in [0,1],$$

with

initial condition	$X(0) = x_0 \in \mathbb{R}$
drift coefficient	$a : [0,1] \times \mathbb{R} \to \mathbb{R}$
diffusion coefficient	$\sigma : [0,1] \times \mathbb{R} \to \mathbb{R}$
(driving) Brownian motion	W (on $(\Omega, \mathcal{A}, P)$)

In this lecture:
Strong (pathwise) approximation of X
i.e.,
Construction of a stochastic process $\widehat{X}$ whose trajectories are close (in a certain sense) to the trajectories of the solution X.

How to interpret (1) ?

Example 1. (population dynamics, finance)

$$dX(t) = \mu \cdot X(t)\, dt + b \cdot X(t)\, dW(t), \quad X(0) = 1.$$

Here: $a(t,x) = \mu \cdot x, \quad \sigma(t,x) = b \cdot x.$

Discrete version:

$$0 = t_0 < t_1 < \ldots < t_n = 1$$

$$\frac{X(t_{\ell+1}) - X(t_\ell)}{X(t_\ell)} = \mu \cdot (t_{\ell+1} - t_\ell) + b \cdot (W(t_{\ell+1}) - W(t_\ell))$$

Average relative increase over time interval $[t_\ell, t_{\ell+1}]$ is proportional to length $t_{\ell+1} - t_\ell$:

$$E\left(\frac{X(t_{\ell+1}) - X(t_\ell)}{X(t_\ell)}\right) = \mu \cdot (t_{\ell+1} - t_\ell)$$

Random perturbation due to, e.g., random change of environment. Modelled by independent noise terms

$$b \cdot (W(t_{\ell+1}) - W(t_\ell)) \sim \mathrm{N}\big(0, b^2 \cdot (t_{\ell+1} - t_\ell)\big)$$

Idealization:

$$\max_\ell (t_{\ell+1} - t_\ell) \to 0$$

Pathwise interpretation of (1) as an ode?

X is a stochastic process with differentiable paths $X(\cdot, \omega),\ \omega \in \Omega$, that satisfy

$$\frac{d}{dt} X(t,\omega) = a(t, X(t,\omega)) + \sigma(t, X(t,\omega)) \cdot \underbrace{\frac{d}{dt} W(t,\omega)}_{?}$$

No! With probability 1, the paths of W are nowhere differentiable on $[0,1]$.

Reformulation of (1) as an integral equation:

$$(2) \quad X(t) = x_0 + \int_0^t a(s, X(s))\, ds + \underbrace{\int_0^t \sigma(s, X(s))\, dW(s)}_{(*)}$$

for $t \in [0,1]$.

Pathwise interpretation of $(*)$ as Riemann-Stieltjes integral?

Problem: With probability 1, the paths of W are of unbounded variation on $[0,1]$.

The way out:

For suitable processes Y define

$$\int_0^t Y(s)\, dW(s)$$

in the mean square sense, and use formulation (2).

The Itô-integral

Let $0 \le s \le t \le 1$ and consider a sequence of discretizations

(3) $$s = t_0^{(n)} < t_1^{(n)} < \ldots < t_n^{(n)} = t$$

with

(4) $$\lim_{n\to\infty} \max_{0\le \ell \le n-1} (t_{\ell+1}^{(n)} - t_\ell^{(n)}) = 0$$

For a NICE process $Y = (Y(u))_{0\le u\le 1}$ define

(5) $$\int_s^t Y(u)\, dW(u)$$

$$:= \operatorname*{l.i.m.}_{n\to\infty} \sum_{\ell=0}^{n-1} Y(t_\ell^{(n)}) \cdot \big(W(t_{\ell+1}^{(n)}) - W(t_\ell^{(n)})\big)$$

(l.i.m.: limit in $L_2(\Omega, \mathcal{A}, P)$)

Remark 1. Formally, NICE means:

(i) Y is adapted (to the standard filtration associated with W)

(ii) Y is measurable

(iii) $\int_0^1 E(Y^2(t))\, dt < \infty$

(iv) Y has continuous paths a.s.

In this lecture: All integrands are NICE

Example 2.

$$\int_s^t W(u)\, dW(u) = 1/2 \cdot \big(W^2(t) - W^2(s) - (t-s)\big)$$

Proof. Consider a sequence of discretizations (3) that satisfies (4). Then

$$S_n := \sum_\ell W(t_\ell^{(n)}) \cdot \big(W(t_{\ell+1}^{(n)}) - W(t_\ell^{(n)})\big)$$
$$= 1/2 \cdot \Big(W^2(t) - W^2(s) - \sum_\ell \big(W(t_{\ell+1}^{(n)}) - W(t_\ell^{(n)})\big)^2\Big).$$

Clearly,

$$E\Big(\sum_\ell \big(W(t_{\ell+1}^{(n)}) - W(t_\ell^{(n)})\big)^2\Big) = t - s.$$

Thus

$$E\Big(\Big(\sum_\ell \big(W(t_{\ell+1}^{(n)}) - W(t_\ell^{(n)})\big)^2 - (t-s)\Big)^2\Big)$$
$$= Var\Big(\sum_\ell \big(W(t_{\ell+1}^{(n)}) - W(t_\ell^{(n)})\big)^2\Big)$$
$$\le \sum_\ell E\Big(\big(W(t_{\ell+1}^{(n)}) - W(t_\ell^{(n)})\big)^4\Big)$$
$$= 3 \cdot \sum_\ell \big(t_{\ell+1}^{(n)} - t_\ell^{(n)}\big)^2$$
$$\le 3 \cdot (t-s) \cdot \max_{0\le \ell\le n-1} (t_{\ell+1}^{(n)} - t_\ell^{(n)}).$$

□

Remark 2. The common rules of Riemann-Stieltjes integration do not hold for the Itô-integral, see Example 2. This is a consequence of the choice of the evaluation points for the process Y in the definition (5).

Different (systematic) choices lead to different stochastic integrals, e.g., the *Stratonovich-integral*

$$\int_s^t Y(u) \circ dW(u)$$

$$:= \operatorname*{l.i.m.}_{n\to\infty} \sum_{\ell=0}^{n-1} Y\big((t_\ell^{(n)} + t_{\ell+1}^{(n)})/2\big) \cdot \big(W(t_{\ell+1}^{(n)}) - W(t_\ell^{(n)})\big),$$

which obeys the classical transformation rules but lacks of other nice properties of the Itô-integral.

Properties of the Itô-integral

(i) Linearity

(ii) For $s \le v \le t$:

$$\int_s^v Y(u)\, dW(u) + \int_v^t Y(u)\, dW(u) = \int_s^t Y(u)\, dW(u)$$

(iii) $E\left(\int_s^t Y(u)\, dW(u)\right) = 0$

(iv) For $s_1 \le s_2 \le t_1 \le t_2$:

$$E\left(\int_{s_1}^{s_2} Y(u)\, dW(u) \cdot \int_{t_1}^{t_2} Y(u)\, dW(u)\right) = 0$$

(v) Itô-isometry:

$$E\left(\left(\int_s^t Y(u)\, dW(u)\right)^2\right) = \int_s^t E(Y^2(u))\, du$$

Furthermore, the process

$$\mathcal{J} = \left(\int_0^t Y(u)\, dW(u)\right)_{0\le t\le 1}$$

satisfies:

(vi) $\mathcal{J}$ is adapted.

(vii) $\mathcal{J}$ is a martingale:

$$E\big(\mathcal{J}(t) \big| W(u), u \le s\big) = \mathcal{J}(s)$$

(viii) $\mathcal{J}$ has a continuous modification $\widetilde{\mathcal{J}}$, i.e.,

$\widetilde{\mathcal{J}}$ has continuous paths and $\forall t:\ \mathcal{J}(t) = \widetilde{\mathcal{J}}(t)$ a.s.

Itô's formula

Theorem 1. *For $f \in C^{1,2}([0,1] \times \mathbb{R})$ and $t \in [0,1]$:*

$$\begin{aligned} f(t, W(t) &= f(0,0) + \int_0^t f^{(0,1)}(s, W(s))\ dW(s) \\ &+ \int_0^t f^{(1,0)}(s, W(s))\ ds \\ &+ 1/2 \cdot \int_0^t f^{(0,2)}(s, W(s))\ ds \end{aligned}$$

Corollary 1. *For $g \in C^1([0,1])$ and $t \in [0,1]$:*

$$\int_0^t g(s)\ dW(s) = g(t) \cdot W(t) - \int_0^t g'(s) \cdot W(s)\ ds$$

Proof. Use Itô's formula with $f(t,x) = g(t) \cdot x$. □

Example 3.

Solving the sde from Example 1

$$dX(t) = \mu \cdot X(t)\ dt + b \cdot X(t)\ dW(t), \quad X(0) = 1.$$

Consider

$$f(t,x) = \exp\big((\mu - b^2/2) \cdot t + b \cdot x\big).$$

Geometric Brownian motion:

$$X(t) := f(t, W(t) = \exp\big((\mu - b^2/2) \cdot t + b \cdot W(t)\big)$$

Note that

$$f^{(1,0)} = (\mu - b^2/2) \cdot f, \quad f^{(0,1)} = b \cdot f, \quad f^{(0,2)} = b^2 \cdot f.$$

Thus, by Itô's formula,

$$\begin{aligned} X(t) &= \underbrace{f(0,0)}_{=1} + \int_0^t b \cdot X(s)\ dW(s) \\ &\quad + \int_0^t (\mu - b^2/2) \cdot X(s)\ ds + 1/2 \cdot \int_0^t b^2 \cdot X(s)\ ds \\ &= 1 + \int_0^t \mu \cdot X(s)\ ds + \int_0^t b \cdot X(s)\ dW(s). \end{aligned}$$

Existence and uniqueness of a solution of (1)

Regularity conditions for a function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$:

(L_f) $\quad \exists K > 0\ \forall x, y \in \mathbb{R}\ \forall t \in [0,1]$:

$$|f(t,x) - f(t,y)| \le K \cdot |x - y|$$

(LG_f) $\quad \exists K > 0\ \forall x \in \mathbb{R}\ \forall t \in [0,1]$:

$$|f(t,x)| \le K \cdot (1 + |x|)$$

(LLG_f) $\exists K > 0\ \forall x \in \mathbb{R}\ \forall s,t \in [0,1]$:

$$|f(s,x) - f(t,x)| \le K \cdot (1 + |x|) \cdot |s - t|$$

Note: $(L_f) \wedge (LLG_f) \Rightarrow (LG_f) \wedge f$ is continuous

Theorem 2. *Assume that $f = a$ and $f = \sigma$ (are continuous and) satisfy (L_f) as well as (LG_f).*
Then there exists a continuous adapted process X that satisfies (2). Moreover, if the process $\widetilde{X}$ has the same properties then

$$P\big(\forall t: \ X(t) = \widetilde{X}(t)\big) = 1.$$

Idea of *proof of existence*:

Consider the Picard-Lindelöf-iteration

$$X^{(0)} = x_0,$$

$$X^{(n)}(t) = x_0 + \int_0^t a\big(s, X^{(n-1)}(s)\big)\, ds + \int_0^t \sigma\big(s, X^{(n-1)}(s)\big)\, dW(s).$$

With probability 1 the paths $X^{(n)}(\cdot,\omega)$ converge uniformly on $[0,1]$. The limiting process is adapted and satisfies (2).

Properties of the solution

Under the assumptions of Theorem 2:

(i) The trajectories of X up to time t are essentially determined by the corresponding trajectories of W up to time t, i.e.,

$$(X(s))_{0 \le s \le t} = g\big((W(s))_{0 \le s \le t}\big) \quad \text{a.s.}$$

for some measurable function $g : C([0,t]) \to C([0,t])$.

(ii) X is Hölder continuous of order $1/2$ in p-th mean sense, i.e.,

$\forall p \in [1, \infty[\ \exists c > 0\ \forall s,t \in [0,1]$:

$$\big(E|X(s) - X(t)|^p\big)^{1/p} \le c \cdot |s - t|^{1/2}. \tag{6}$$

(iii) X is a Markov-process. For $s \le t$:

$$P^{X(t)|(X(u))_{0 \le u \le s}} = P^{X(t)|X(s)}$$

Moreover, for $P^{X(s)}$ almost all $x \in \mathbb{R}$:

$$P^{X(t)|X(s)=x} = P^{X_{s,x}(t)},$$

where $X_{s,x}$ is the solution of

$$dX_{s,x}(t) = a(t, X_{s,x})\, dt + \sigma(t, X_{s,x})\, dW(t), \quad t \in [s,1],$$

with initial value $X_{s,x}(s) = x$.

(iv) X is a diffusion process. In particular,

$$E(X(t+\delta) - X(t)|X(t) = x) = a(t,x) \cdot \delta + o(\delta)$$

and

$$E\big((X(t+\delta) - X(t))^2 \big| X(t) = x\big) = \sigma^2(t,x) \cdot \delta + o(\delta). \tag{7}$$

Property (7) is crucial for the analysis of L_2-approximation, see Part III. The smoothness of X in mean square sense is locally in time and space determined by:

conditional Hölder constant $|\sigma(t, X(t))|$.

Example 4. For the trivial equation

$$dX(t) = dW(t), \quad X(0) = 0,$$

we have

$$\sigma(t, X(t) = 1.$$

For the linear equation

$$dX(t) = \mu \cdot X(t)\; dt + b \cdot X(t)\; dW(t), \quad X(0) = 1.$$

we have

$$\sigma(t, X(t) = b \cdot X(t) = b \cdot \exp\big((\mu - b^2/2) \cdot t + b \cdot W(t)\big),$$

see Example 3.

The problem of strong approximation

Given data about a, σ and W construct a stochastic process $\widehat{X}$ that is pathwise close to the solution X of (1).

Available data:

(i) evaluation of a, σ and/or partial derivatives of these functions at suitably chosen points in $[0,1] \times \mathbb{R}$, e.g.,

$$a(t,x),\ \sigma(t,x),\ a^{(0,1)}(t,x),\ \sigma^{(2,0)}(t,x).$$

In practice: by subroutines

(ii) evaluation of suitably chosen functionals of W; in particular

(a) Dirac-functionals: $W(t)$,

(b) Continuous linear functionals: $\int_s^t W(u)\; du$,

(c) Iterated Itô-integrals: $\int_s^t W(u)\; dW(u)$.

In practice: (a), (b) by random number generator;
(c) is a problem.

Discrete approximation:

$\widehat{X}$ pathwise close to X at finitely many points in $[0,1]$, e.g., at $t=1$ with

error pathwise: $|X(1) - \widehat{X}(1)|$

average: $\big(E|X(1) - \widehat{X}(1)|^q\big)^{1/q}, \quad q \in [1, \infty[.$

Global approximation:

$\widehat{X}$ pathwise close to X globally on $[0,1]$ with

error pathwise:

$$\|X-\widehat{X}\|_p = \begin{cases} \left(\int_0^1 |X(t)-\widehat{X}(t)|^p\,dt\right)^{1/p} & , p\in[1,\infty[\\ \sup_{t\in[0,1]} |X(t)-\widehat{X}(t)| & , p=\infty. \end{cases}$$

average: $\left(E\|X-\widehat{X}\|_p^q\right)^{1/q}, \quad q\in[1,\infty[.$

Remark 3. Closely related problems:

- Reconstruction of a stochastic process X
- Estimation of a weighted integral of X

based on data about X itself. See Ritter (2000), Remark 10 in Part III and Remark 14 in Part IV.

Note: For approximation of the solution X of (1) only data about the driving Brownian motion W is available.

II. Classical methods and results

Classical methods are based on a fixed discretization

$$0=t_0<t_1<\ldots<t_n=1.$$

Recursive one-step construction of a scheme

$$x_0=\widehat{X}(t_0), \widehat{X}(t_1), \ldots, \widehat{X}(t_n).$$

Discrete approximation at $t=1$: $\widehat{X}(1)$.

Global approximation by (piecewise linear) interpolation.

The Euler scheme (Maruyama 1955)

Linearization of integrals:

$$X(t_{\ell+1}) = X(t_\ell) + \underbrace{\int_{t_\ell}^{t_{\ell+1}} a(t,X(t))\,dt}_{(*)} + \underbrace{\int_{t_\ell}^{t_{\ell+1}} \sigma(t,X(t))\,dW(t)}_{(**)}$$

$$\begin{array}{lll} (*) & \text{approx} & a(t_\ell,X(t_\ell))\cdot(t_{\ell+1}-t_\ell), \\ (**) & \text{approx} & \sigma(t_\ell,X(t_\ell))\cdot(W(t_{\ell+1})-W(t_\ell)). \end{array}$$

Scheme:

$$\widehat{X}^E(t_0) = x_0,$$
$$\widehat{X}^E(t_{\ell+1}) = \widehat{X}^E(t_\ell) + a(t_\ell, \widehat{X}^E(t_\ell)) \cdot (t_{\ell+1} - t_\ell) + \sigma(t_\ell, \widehat{X}^E(t_\ell)) \cdot (W(t_{\ell+1}) - W(t_\ell)).$$

Example 5.

Trivial equation:

$$dX(t) = dW(t), \quad X(0) = 0$$
$$X = W$$
$$\widehat{X}^E(t_\ell) = W(t_\ell)$$

Geometric Brownian motion:

$$dX(t) = \mu \cdot X(t)\, dt + b \cdot X(t)\, dW(t), \quad X(0) = 1$$
$$X(t) = \exp\big((\mu - b^2/2) \cdot t + b \cdot W(t)\big)$$
$$a(t,x) = \mu \cdot x$$
$$\sigma(t,x) = b \cdot x$$

Here

$$\widehat{X}^E(t_{\ell+1}) = \widehat{X}^E(t_\ell) + \mu \cdot \widehat{X}^E(t_\ell) \cdot (t_{\ell+1} - t_\ell) + b \cdot \widehat{X}^E(t_\ell) \cdot (W(t_{\ell+1}) - W(t_\ell))\big)$$

and

$$\widehat{X}^E(t_\ell) = \prod_{j=0}^{\ell-1} \big(1 + \mu \cdot (t_{j+1} - t_j) + b \cdot (W(t_{j+1}) - W(t_j))\big).$$

Notation:

$$\Delta_{\max} = \max_{\ell=0,\dots,n-1} (t_{\ell+1} - t_\ell)$$

c: unspecified positive constants that only depend on error parameters p, q, on initial value x_0, and on constants from regularity conditions on a and σ

Theorem 3. *Assume* (L_f) *and* (LLG_f) *for* $f = a$ *and* $f = \sigma$.

(i) *(Maruyama 1955)*

$$\max_\ell \big(E|X(t_\ell) - \widehat{X}^E(t_\ell)|^q\big)^{1/q} \le c \cdot \Delta_{\max}^{1/2}$$

(ii) *For the piecewise linear interpolated Euler scheme*

$$\big(E\|X - \widehat{X}^E\|_p^q\big)^{1/q} \le c \cdot \begin{cases} \Delta_{\max}^{1/2} & \text{if } p < \infty \\ \Delta_{\max}^{1/2} \cdot \big(\ln \Delta_{\max}^{-1}\big)^{1/2} & \text{if } p = \infty \\ \text{(Fauré 1992)} & \end{cases}$$

Proof of the first inequality in Theorem 3(ii). Define

$$X^{lin}(t) = \frac{t_{\ell+1} - t}{t_{\ell+1} - t_\ell} \cdot X(t_\ell) + \frac{t - t_\ell}{t_{\ell+1} - t_\ell} \cdot X(t_{\ell+1}), \quad t \in [t_{\ell+1}, t_\ell].$$

By property (6) and part (i) of Theorem 3

$$\big(E\|X - X^{lin}\|_p^q\big)^{1/q} \le c \cdot \Delta_{\max}^{1/2},$$
$$\big(E\|X^{lin} - \widehat{X}^E\|_p^q\big)^{1/q} \le c \cdot \max_\ell \big(E|X(t_\ell) - \widehat{X}^E(t_\ell)|^q\big)^{1/q} \le c \cdot \Delta_{\max}^{1/2}.$$

□

The Milstein scheme (Milstein 1974)

Motivation: Assumptions in Theorem 3 on a and σ imply

$$E\left(\int_{t_\ell}^{t_{\ell+1}} a(s,X(s))\,ds - a(t_\ell,X(t_\ell))\cdot(t_{\ell+1}-t_\ell)\right)^2 \le c\cdot(t_{\ell+1}-t_\ell)^3$$

but only

$$E\left(\int_{t_\ell}^{t_{\ell+1}} \sigma(s,X(s))\,dW(s) - \sigma(t_\ell,X(t_\ell))\cdot\big(W(t_{\ell+1})-W(t_\ell)\big)\right)^2 \le c\cdot(t_{\ell+1}-t_\ell)^2$$

A better approximation of the Itô-integral is needed.

Idea: Use first order Taylor-expansion of $\sigma(t,\cdot)$ and ignore $O\big((t_{\ell+1}-t_\ell)^3\big)$-terms (in mean square sense).

Under sufficient regularity of a and σ for $t\in[t_\ell,t_{\ell+1}]$

$$\begin{aligned}
\sigma(t,X(t)) &\overset{\text{m.s.}}{=} \sigma(t_\ell,X(t_\ell)) + \sigma^{(0,1)}(t_\ell,X(t_\ell))\cdot(X(t)-X(t_\ell)) \\
&\quad + O\big((t_{\ell+1}-t_\ell)^2\big) \\
&\overset{\text{m.s.}}{=} \sigma(t_\ell,X(t_\ell)) + \sigma^{(0,1)}(t_\ell,X(t_\ell))\cdot\int_{t_\ell}^{t}\sigma(s,X(s))\,dW(s) \\
&\quad + O\big((t_{\ell+1}-t_\ell)^2\big) \\
&\overset{\text{m.s.}}{=} \sigma(t_\ell,X(t_\ell)) + \big(\sigma^{(0,1)}\cdot\sigma\big)(t_\ell,X(t_\ell))\cdot\big(W(t)-W(t_\ell)\big) \\
&\quad + O\big((t_{\ell+1}-t_\ell)^2\big),
\end{aligned}$$

which yields

$$\begin{aligned}
&\int_{t_\ell}^{t_{\ell+1}} \sigma(t,X(t))\,dW(t) \\
&\quad\overset{\text{m.s.}}{=} \sigma(t_\ell,X(t_\ell))\cdot\big(W(t_{\ell+1})-W(t_\ell)\big) \\
&\qquad + \big(\sigma^{(0,1)}\cdot\sigma\big)(t_\ell,X(t_\ell))\cdot\int_{t_\ell}^{t_{\ell+1}}\big(W(t)-W(t_\ell)\big)\,dW(t) \\
&\qquad + O\big((t_{\ell+1}-t_\ell)^3\big).
\end{aligned}$$

Recall from Example 2:

$$\int_{t_\ell}^{t_{\ell+1}}\big(W(t)-W(t_\ell)\big)\,dW(t) = 1/2\cdot\big((W(t_{\ell+1})-W(t_\ell))^2-(t_{\ell+1}-t_\ell)\big)$$

Scheme:

$$\begin{aligned}
\widehat{X}^M(t_0) &= x_0, \\
\widehat{X}^M(t_{\ell+1}) &= \widehat{X}^M(t_\ell) + a(t_\ell,\widehat{X}^M(t_\ell))\cdot(t_{\ell+1}-t_\ell) \\
&\quad + \sigma(t_\ell,\widehat{X}^M(t_\ell))\cdot(W(t_{\ell+1})-W(t_\ell)) \\
&\quad + 1/2\cdot\big(\sigma^{(0,1)}\cdot\sigma\big)(t_\ell,\widehat{X}^M(t_\ell)) \\
&\qquad \times\big((W(t_{\ell+1})-W(t_\ell))^2-(t_{\ell+1}-t_\ell)\big).
\end{aligned}$$

Example 6. Geometric Brownian motion

$$\begin{aligned} dX(t) &= \mu \cdot X(t)\, dt + b \cdot X(t)\, dW(t), \quad X(0) = 1 \\ X(t) &= \exp\big((\mu - b^2/2) \cdot t + b \cdot W(t)\big) \end{aligned}$$

Here: $\sigma^{(0,1)} = b$

Hence

$$\begin{aligned} \widehat{X}^M(t_{\ell+1}) = \widehat{X}^M(t_\ell) &+ \mu \cdot \widehat{X}^M(t_\ell) \cdot (t_{\ell+1} - t_\ell) \\ &+ b \cdot \widehat{X}^M(t_\ell) \cdot (W(t_{\ell+1}) - W(t_\ell))\big) \\ &+ b^2/2 \cdot \widehat{X}^M(t_\ell) \\ &\times \big((W(t_{\ell+1}) - W(t_\ell))^2 - (t_{\ell+1} - t_\ell)\big) \end{aligned}$$

and

$$\begin{aligned} \widehat{X}^M(t_\ell) = \prod_{j=0}^{\ell-1} \Big(1 &+ (\mu - b^2/2) \cdot (t_{j+1} - t_j) \\ &+ b \cdot (W(t_{j+1}) - W(t_j)) \\ &+ b^2/2 \cdot (W(t_{j+1}) - W(t_j))^2\Big). \end{aligned}$$

Theorem 4. *Assume* (L_f), (LLG_f) *and* $(L_{f^{(0,1)}})$ *for* $f = a$ *and* $f = \sigma$.

(i) *(Milstein 1974)*

$$\max_\ell \big(E|X(t_\ell) - \widehat{X}^M(t_\ell)|^q\big)^{1/q} \le c \cdot \Delta_{\max}$$

(ii) *For the piecewise linear interpolated Milstein scheme*

$$\begin{aligned} &\big(E\|X - \widehat{X}^M\|_p^q\big)^{1/q} \\ &\le c \cdot \begin{cases} \Delta_{\max}^{1/2} & \textit{if } p < \infty \\ \Delta_{\max}^{1/2} \cdot \big(\ln \Delta_{\max}^{-1}\big)^{1/2} & \textit{if } p = \infty \end{cases} \end{aligned}$$

Proof of Theorem 4(ii). The first inequality is straightforward from property (6) and part (i); c.f. the proof of the first inequality in Theorem 3(ii). The second inequality follows from Theorem 3 and Theorem 4(i) since

$$\begin{aligned} \big(E\|X - \widehat{X}^M\|_\infty^q\big)^{1/q} \le \big(E\|X - \widehat{X}^E\|_\infty^q\big)^{1/q} &+ \max_\ell \big(E|\widehat{X}^E(t_\ell) - X(t_\ell)|^q\big)^{1/q} \\ &+ \max_\ell \big(E|X(t_\ell) - \widehat{X}^M(t_\ell)|^q\big)^{1/q}. \quad \square \end{aligned}$$

See ?? for a proof of Theorem 4(i).

Can we do better?

Discrete:	NO	if only point evaluations of W are used see Part IV
	YES	if continuous linear functionals of W or iterated Itô-integrals are used, see next section
Global:	NO	see Part III and Part V

Itô-Taylor schemes (Kloeden and Platen 1995)

Iterated Itô-integrals:

Notation $\quad d_0 t := dt,\ d_1 t := dW(t)$

Let $\quad \mathcal{M} = \bigcup_{r\in\mathbb{N}} \{0,1\}^r$

For $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathcal{M}$ and $0 \le s \le t \le 1$ define

$$I_{\alpha,s,t} = \int_s^t \int_s^{t_r} \cdots \int_s^{t_2} 1\ d_{\alpha_1} t_1 \cdots d_{\alpha_{r-1}} t_{r-1}\, d_{\alpha_r} t_r.$$

Example 7.

$$I_{(0),s,t} = \int_s^t 1\ du = t - s$$

$$I_{(1),s,t} = \int_s^t 1\ dW(u) = W(t) - W(s)$$

$$I_{(1,0),s,t} = \int_s^t I_{(1),s,u}\ du = \int_s^t (W(u) - W(s))du$$

(continuous linear functional)

$$I_{(1,0,1),s,t} = \int_s^t \left(\int_s^v (W(u) - W(s))\ du \right)\ dW(v)$$

(nonlinear functional)

Note: In general, iterated Itô-integrals are not Gaussian.

Itô-Taylor scheme of order $\gamma \in \mathbb{N}/2$:

$$\widehat{X}^\gamma(t_0) = x_0$$

$$\widehat{X}^\gamma(t_{\ell+1}) = \widehat{X}^\gamma(t_\ell) + \sum_{\alpha\in A_\gamma} f_\alpha\big(t_\ell, \widehat{X}^\gamma(t_\ell)\big) \cdot I_{\alpha,t_\ell,t_{\ell+1}}$$

with certain hierarchical sets of multiindices $A_\gamma \subset \mathcal{M}$ and certain coefficient functions

$$f_\alpha : [0,1] \times \mathbb{R} \to \mathbb{R}$$

composed of a, σ (and partial derivatives).

(Background: Truncated Itô-Taylor expansion of solution X)

Coefficient functions:

Consider the differential operators

$$L^0 = \frac{\partial}{\partial t} + a \cdot \frac{\partial}{\partial x} + \sigma^2/2 \cdot \frac{\partial^2}{\partial x^2}, \quad L^1 = \sigma \cdot \frac{\partial}{\partial x}$$

For $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathcal{M}$ define

$$f_\alpha = \begin{cases} a & \text{if } \alpha = (0) \\ \sigma & \text{if } \alpha = (1) \\ L^{\alpha_1} f_{(\alpha_2,\ldots,\alpha_r)} & \text{otherwise} \end{cases}$$

Example 8.

$$f_{(0,1)} = \sigma^{(1,0)} + a \cdot \sigma^{(0,1)} + \sigma^2/2 \cdot \sigma^{(0,2)}$$

$$f_{(1,0)} = \sigma \cdot a^{(0,1)}, \quad f_{(1,1)} = \sigma \cdot \sigma^{(0,1)}$$

Hierarchical sets:

For $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathcal{M}$ put

$$\zeta(\alpha) = \sharp\{j :\ \alpha_j = 0\}, \quad ||\alpha|| = r + \zeta(\alpha)$$

For $\gamma \in \mathbb{N}/2$ define

$$A_\gamma = \{\alpha \in \mathcal{M} :\ ||\alpha|| \le 2\gamma \text{ or } ||\alpha|| = 2\zeta(\alpha) = 2\gamma + 1\}$$

Note: $\gamma_1 < \gamma_2 \Rightarrow A_{\gamma_1} \subset A_{\gamma_2}$

Example 9.

$\gamma = 1/2$: $A_{1/2} = \{(0), (1)\}$
$f_{(0)} = a,\ f_{(1)} = \sigma$
Euler scheme

$\gamma = 1$: $A_1 = \{(0), (1), (1,1)\}$
$f_{(0)} = a,\ f_{(1)} = \sigma,\ f_{(1,1)} = \sigma \cdot \sigma^{(0,1)}$
Milstein scheme

$\gamma = 3/2$: $A_{3/2} = \{(0), (1), (0,0), (0,1), (1,0), (1,1), (1,1,1)\}$
yields the

Wagner-Platen scheme (Wagner and Platen 1978)

$$\widehat{X}^{3/2}(t_0) = x_0,$$

$$\begin{aligned}
\widehat{X}^{3/2}&(t_{\ell+1}) \\
&= \widehat{X}^{3/2}(t_\ell) + a(t_\ell, \widehat{X}^{3/2}(t_\ell)) \cdot (t_{\ell+1} - t_\ell) \\
&\qquad + \sigma(t_\ell, \widehat{X}^{3/2}(t_\ell)) \cdot (W(t_{\ell+1}) - W(t_\ell)) \\
&\quad + 1/2 \cdot \big(\sigma^{(0,1)} \cdot \sigma\big)(t_\ell, \widehat{X}^{3/2}(t_\ell)) \\
&\qquad \times \big((W(t_{\ell+1}) - W(t_\ell))^2 - (t_{\ell+1} - t_\ell)\big) \\
&\quad + \big(\sigma^{(1,0)} + a \cdot \sigma^{(0,1)} - \sigma/2 \cdot (\sigma^{(0,1)})^2\big)(t_\ell, \widehat{X}^{3/2}(t_\ell)) \\
&\qquad \times \big(W(t_{\ell+1}) - W(t_\ell)\big) \cdot (t_{\ell+1} - t_\ell) \\
&\quad + 1/6 \cdot \big(\sigma \cdot (\sigma^{(0,1)})^2 + \sigma^2 \cdot \sigma^{(0,2)}\big)(t_\ell, \widehat{X}^{3/2}(t_\ell)) \\
&\qquad \times \big(W(t_{\ell+1}) - W(t_\ell)\big)^3 \\
&\quad + 1/2 \cdot \big(a^{(1,0)} + a \cdot a^{(0,1)} + \sigma^2/2 \cdot a^{(0,2)}\big)(t_\ell, \widehat{X}^{3/2}(t_\ell)) \\
&\qquad \times (t_{\ell+1} - t_\ell)^2 \\
&\quad + \big(\sigma \cdot a^{(0,1)} - \sigma^{(1,0)} \\
&\qquad\qquad - a \cdot \sigma^{(0,1)} - \sigma^2/2 \cdot \sigma^{(0,2)}\big)(t_\ell, \widehat{X}^{3/2}(t_\ell)) \\
&\qquad \times \int_{t_\ell}^{t_{\ell+1}} \big(W(s) - W(t_\ell)\big)\, ds
\end{aligned}$$

Note: The Wagner-Platen scheme uses point evaluations <u>and</u> continuous linear functionals of W.

For $\gamma \in \mathbb{N}/2$ put

$$B_\gamma = \{(\alpha_1, \dots, \alpha_r) \in \mathcal{M} \setminus A_\gamma : \ (\alpha_2, \dots, \alpha_r) \in A_\gamma\}$$

Theorem 5. *Assume* (L_{f_α}) *and* (LLG_{f_α}) *for all* $\alpha \in A_\gamma \cup B_\gamma$.

(i) *(Kloeden and Platen 1995)*

$$\max_\ell \left(E|X(t_\ell) - \widehat{X}^\gamma(t_\ell)|^q\right)^{1/q} \le c \cdot \Delta_{\max}^\gamma$$

(ii) *For the piecewise linear interpolated scheme* $\widehat{X}^\gamma$

$$\left(E\|X - \widehat{X}^\gamma\|_p^q\right)^{1/q} \le c \cdot \begin{cases} \Delta_{\max}^{1/2} & \text{if } p < \infty \\ \Delta_{\max}^{1/2} \cdot \left(\ln \Delta_{\max}^{-1}\right)^{1/2} & \text{if } p = \infty \end{cases}$$

Remark 4. Part (ii) of Theorem 5 is shown in the same manner as Theorem 4(ii).

Note again: In general, the simulation of iterated Itô-integrals is an unsolved problem. See, e.g., Gaines and Lyons (1994), and Rydén and Wiktorsson (2001) for that problem.

Remark 5. Meanwhile there exists a multitude of schemes including, e.g., Runge-Kutta methods and multistep methods. See Kloeden and Platen (1995).

II. Optimal L_2-approximation based on point evaluations of W

Motivation:

Consider a classical strong approximation $\widehat{X}$ based on a discretization $0 = t_0 < \dots < t_n = 1$. Typical result:

$$\left(E\|X - \widehat{X}\|_2^2\right)^{1/2} \le c \cdot \Delta_{\max}^{1/2}$$

(e.g., Theorems 3, 4 and 5 for $p = q = 2$ in Part II)

Shortcomings of this kind of result are

(i) Computational cost of $\widehat{X}$ is not taken into account, e.g.,

(a) number of calls to random number generator

(b) number of evaluations of a, σ, $a^{(0,1)}$, etc.

(c) number of arithmetic operations to compute $\widehat{X}$

A more useful result would be of the type

$$\left(E\|X - \widehat{X}\|_2^2\right)^{1/2} \le c \cdot \left(\mathrm{cost}(\widehat{X})\right)^{-1/2}$$

with an appropriate definition of $\mathrm{cost}(\widehat{X})$, e.g.,

$$\mathrm{cost}(\widehat{X}) = (a) + (b) + (c)$$

$$\text{or} \quad \mathrm{cost}(\widehat{X}) = (a).$$

Thus: Relate the error to the cost

(ii) The result gives only an upper bound with an unspecified constant. Different methods or different discretizations can not be compared.

Thus: Lower bounds and explicit constants

Questions:

- Optimal method?
- Adaptive discretization?
- Complexity?

  Given $\varepsilon > 0$ what is the minimal cost to spend in order to achieve an error of at most ε?

  or, equivalently,

  Given a (cost)-budget N what is the minimal error that can be achieved using methods with cost at most N?

For a systematic treatment of these questions we need to specify

- class of methods
- error criterion
- cost measure

(Approach in the framework of *information based complexity*, see Traub, Wasilkowski and Woźniakowski 1988)

Class of methods

$\mathbb{X}$: All (up to measurability) approximations $\widehat{X}$ that are are based on a finite number of evaluations of W at points

$$\tau_1, \ldots, \tau_\nu \in [0,1]$$

where the choice of $\tau_{\ell+1}$ may depend in any (measurable) way on the previously computed values $W(\tau_1), \ldots, W(\tau_\ell)$ (in particular $\tau_\ell < \tau_{\ell+1}$ is not required).

and the total number ν of evaluations may be determined by a (measurable) stopping rule.

(See below for a formal definition of $\mathbb{X}$)

Error

$$e(\widehat{X}) = \big(E\|X - \widehat{X}\|_2^2\big)^{1/2} = \left(E \int_0^1 \big(X(t) - \widehat{X}(t)\big)^2 \, dt\right)$$

Cost

$$c(\widehat{X}) = E(\nu) = \text{average number of calls to random number generator}$$

Minimal errors

$$e_N = \inf\{e(\widehat{X}) : \ \widehat{X} \in \mathbb{X},\ c(\widehat{X}) \le N\}, \quad N > 0$$

The quantity e_N is the minimal L_2-error that can be achieved with methods in the class $\mathbb{X}$ that use at most N point evaluations of W on the average.

Mathematical goals:

(i) Exact rate of convergence of e_N, i.e., order and asymptotic constant (answers complexity question in asymptotic sense)

(ii) Easy to implement method $\widehat{X}_n$ that achieves this rate, i.e.,

$$\lim_{n\to\infty} \frac{e(\widehat{X}_n)}{e_{c(\widehat{X}_n)}} = 1$$

((strong) asymptotic optimality)

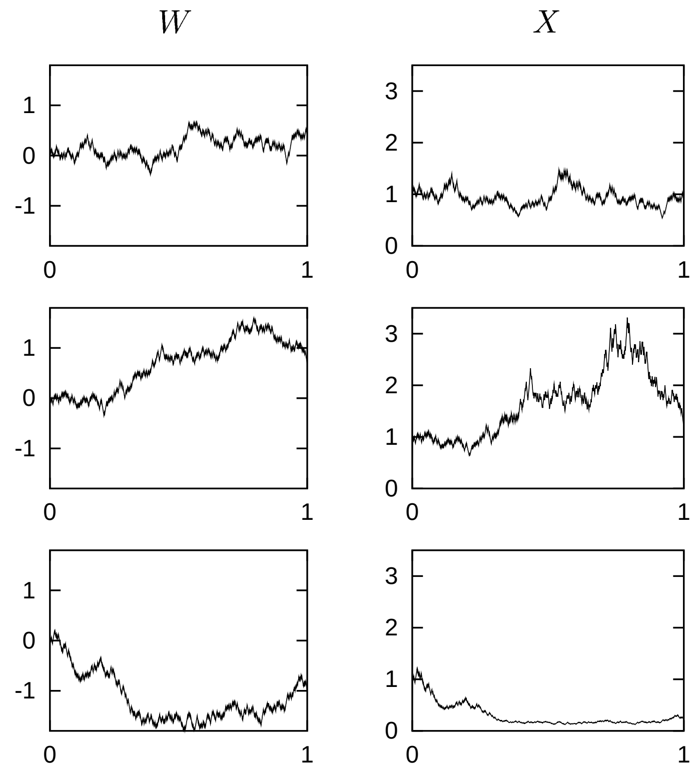
Remark 6. The class $\mathbb{X}$ is specified by the type of information about W. Here only Dirac functionals are admitted. In particular, $\mathbb{X}$ contains (piecewise) interpolated Euler and Milstein schemes but not interpolated Itô-Taylor schemes of order $\gamma > 1$. See Part V for results concerning continuous linear functionals or iterated Itô-integrals.

Why should the points $\tau_1, \ldots, \tau_\nu$ be chosen sequentially?

Example 10. (Geometric Brownian motion)

$$dX(t) = X(t)\, dW(t), \quad X(0) = 1$$
$$X(t) = \exp(-t/2 + W(t))$$

Three trajectories of W and X:



Discretization should be adapted to trajectories of solution

Recall from Part I:

Under appropriate conditions on a and σ

$$E\big((X(t+\delta)-X(t))^2 \big| X(t)=x\big) = \sigma^2(t,x)\cdot\delta + o(\delta)$$

Oszillation of trajectories of X is (in mean square sense) locally in time and space determined by:

conditional Hölder constant $|\sigma(t, X(t))|$

In the preceding example: $|\sigma(t, X(t))| = X(t)$
(c.f. Example 4)

Basic idea: The *larger* the value of $|\sigma(t, X(t))|$
the *finer* the discretization around t.

Problems:

(i) Quantification of *larger* and *finer*

(ii) We can not observe the value of $|\sigma(t, X(t))|$ in general

The fully adaptive method $\widehat{X}_n^*$

Start: $\tau_0 = 0, \quad \widehat{X}_n^*(\tau_0) = x_0$

ℓ-th step:

- known data
$$\tau_j,\ x_j = \widehat{X}_n^*(\tau_j), \quad j = 0,\ldots,\ell-1$$
- estimate the conditional Hölder constant at $\tau_{\ell-1}$ by
$$|\sigma(\tau_{\ell-1}, x_{\ell-1})|$$
and compute candidate for next discretization point
$$\tau_\ell = \tau_{\ell-1} + \frac{1}{n\cdot|\sigma(\tau_{\ell-1}, x_{\ell-1})|}$$
- If $\tau_\ell < 1$ then do Milstein step
$$\begin{aligned}\widehat{X}_n^*(\tau_\ell) = x_{\ell-1} &+ a(\tau_{\ell-1}, x_{\ell-1}))\cdot(\tau_\ell - \tau_{\ell-1}) \\ &+ \sigma(\tau_{\ell-1}, x_{\ell-1}))\cdot(W(\tau_\ell) - W(\tau_{\ell-1})) \\ &+ 1/2\cdot\big(\sigma\cdot\sigma^{0,1)}\big)(\tau_{\ell-1}, x_{\ell-1})) \\ &\quad\times\big((W(\tau_\ell) - W(\tau_{\ell-1}))^2 - (\tau_\ell - \tau_{\ell-1})\big)\end{aligned}$$
and proceed with $(\ell+1)$-th step.
Else set $\tau_\ell = 1$, calculate $\widehat{X}_n^*(1)$ as above and stop

End: Piecewise linear interpolation

Number of evaluations of W pathwise

$$\nu_n = \min\{\ell \in \mathbb{N} : \ \tau_\ell = 1\}$$
$$\text{approx } n \cdot \int_0^1 |\sigma(t, X(t))| \, dt$$

The *harder* the trajectory the more points are used

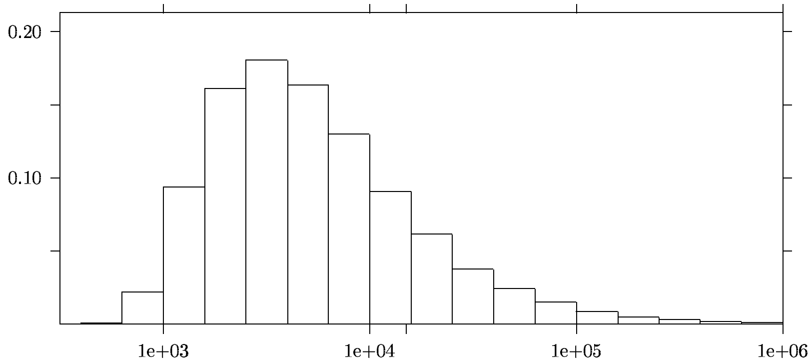
Simulation experiment:

$$dX(t) = 3X(t)\, dW(t)$$
$$n = 5000$$

10000 trajectories $w_1, \dots, w_{10000}$ of W
Average number of evaluations

$$\frac{1}{10^4} \sum_{i=1}^{10^4} \nu_{5000}(w_i) = 14981$$

Relative frequencies of ν_{5000}:



Unfortunately, it's hard to analyze $\widehat{X}_n^*$

The simplified adaptive method $\widehat{X}_n$

Idea: Do not update the step-size in every step

Algorithm:

1) Dependent on n determine *coarse* equidistant prediscretization
$$t_\ell = \ell / k_n, \quad \ell = 0, 1, \dots, k_n$$
Long Milstein-steps
$$\widehat{X}_n(t_\ell) = \widehat{X}^M(t_\ell)$$

2) Determine number of additional equidistant points in subintervals $]t_\ell, t_{\ell+1}[$ by
$$\mu_\ell = \lceil n/k_n \cdot |\sigma(t_\ell, \widehat{X}^M(t_\ell))| \rceil$$
Resulting points
$$\tau_{\ell,j} = t_\ell + j \cdot (k_n \cdot (\mu_\ell + 1))^{-1}, \quad j = 0, 1, \dots, \mu_\ell$$
Short Euler-steps with "frozen" values of drift and diffusion coefficient
$$\widehat{X}_n(\tau_{\ell,j}) = \widehat{X}_n(\tau_{\ell,j-1}) + a(t_\ell, \widehat{X}^M(t_\ell)) \cdot (\tau_{\ell,j} - \tau_{\ell,j-1}) + \sigma(t_\ell, \widehat{X}^M(t_\ell)) \cdot (W(\tau_{\ell,j}) - W(\tau_{\ell,j-1}))$$

3) Piecewise linear interpolation

Adjusting the number k_n of coarse grid points

k_n should be small compared to the total number $\sum_\ell \mu_\ell$ of adaptive points in order to keep track of the local smoothness of the solution:

$$\lim_{n\to\infty} k_n/n = 0$$

k_n should be large enough to guarantee sufficiently good estimates of the values of the drift and diffusion coefficients:

$$\lim_{n\to\infty} k_n^2/n = \infty$$

Assumptions throughout the rest of Part III:

$$(L_f),\ (LLG_f),\ (L_{f(0,1)}) \quad \text{for } f = a \text{ and } f = \sigma$$

Put

$$C = \frac{1}{\sqrt{6}} \cdot E \int_0^1 |\sigma(t, X(t))|\, dt$$

Theorem 6. *(Hofmann, M-G, Ritter 2001)*

(i) $$\lim_{n\to\infty} \big(c(\widehat{X}_n)\big)^{1/2} \cdot e(\widehat{X}_n) = C$$

(ii) $$\lim_{n\to\infty} n^{-1} \cdot c(\widehat{X}_n) = \sqrt{6} \cdot C.$$

A first comparison:

$\widehat{X}_n^M$: Piecewise linear interpolated Milstein scheme with equidistant discretization $0, 1/n, 2/n, \ldots, 1$

Note: $c(\widehat{X}_n^M) = n$

Put

$$C^{\text{equi}} = \frac{1}{\sqrt{6}} \cdot \left(E \int_0^1 \sigma^2(t, X(t))\, dt\right)^{1/2}$$

Theorem 7. *(Hofmann, M-G, Ritter 2001)*

$$\lim_{n\to\infty} n^{1/2} \cdot e(\widehat{X}_n^M) = C^{equi}$$

Example 11. Geometric Brownian motion

$$dX(t) = b \cdot X(t)\, dW(t), \quad X(0) = 1$$

Here

$$C^{\text{equi}} = \frac{1}{\sqrt{6}} \cdot \big(\exp(b^2) - 1\big)^{1/2}, \quad C = \frac{1}{\sqrt{6}} \cdot |b|$$

<u>The class $\mathbb{X}$</u>

An approximation $\widehat{X} \in \mathbb{X}$ is determined by 3 sequences of measurable mappings

$$\psi = (\psi_k)_{k\in\mathbb{N}},\ \chi = (\chi_k)_{k\in\mathbb{N}},\ \varphi = (\varphi_k)_{k\in\mathbb{N}},$$

with

$$\begin{aligned} &\psi_\ell : \mathbb{R}^\ell \to [0,1] && \text{(determines } \tau_\ell\text{)}, \\ &\chi_\ell : \mathbb{R}^{\ell+1} \to \{\text{STOP}, \text{GO}\} && \text{(determines stopping)}, \\ &\varphi_\ell : \mathbb{R}^{\ell+1} \to L_2([0,1]) && \text{(determines approximation)}. \end{aligned}$$

First evaluation point: $\tau_1 = \psi_1(x_0)$

ℓ-th step: known data

$$D_\ell = (x_0, W(\tau_1), \ldots, W(\tau_\ell))$$

if $\chi_\ell(D_\ell) = \text{GO}$ then evaluate W at

$$\tau_{\ell+1} = \psi_{\ell+1}(D_\ell)$$

if $\chi_\ell(D_\ell) = \text{STOP}$ then use approximation

$$\varphi_\ell(D_\ell)$$

Total number of evaluations:

$$\nu_{(\psi,\chi,\varphi)} = \min\{\ell \in \mathbb{N} : \chi_\ell(D_\ell) = \text{STOP}\}$$

Put

$$S = \{(\psi,\chi,\varphi) : P(\nu_{(\psi,\chi,\varphi)} < \infty) = 1\}$$

For $(\psi,\chi,\varphi) \in S$ define

$$\widehat{X}_{(\psi,\chi,\varphi)} = \varphi_{\nu_{(\psi,\chi,\varphi)}}\big(D_{\nu_{(\psi,\chi,\varphi)}}\big)$$

andlet

$$\mathbb{X} = \big\{\widehat{X}_{(\psi,\chi,\varphi)} : (\psi,\chi,\varphi) \in S\big\}$$

Note:

- $\mathbb{X}$ contains every implementable method;
  in particular all methods based on evaluation of W at a fixed discretization
  $$0 = t_0 < \ldots < t_n = 1.$$
  In this case
  $$\psi_\ell \equiv t_\ell, \quad \ell = 1, \ldots, n,$$
  $$\chi_\ell \equiv \begin{cases} \text{GO} & \text{if } \ell \le n-1 \\ \text{STOP} & \text{if } \ell = n \end{cases}$$
- $\mathbb{X}$ contains the adaptive methods $\widehat{X}_n$ and $\widehat{X}_n^*$
- $\mathbb{X}$ even contains methods, which may not be implementable, e.g., conditional expectations
  $$\widehat{X} = E(X \mid W(\tau_1), \ldots, W(\tau_\nu))$$
- No restrictions on information about a and σ.

Recall minimal errors $\quad e_N = \inf\big\{e(\widehat{X}) : \widehat{X} \in \mathbb{X},\ c(\widehat{X}) \le N\big\}$

Theorem 8. *(Hofmann, M-G, Ritter 2001)*

$$\lim_{N\to\infty} N^{1/2} \cdot e_N = C$$

Corollary 2. *If $C > 0$ then $\widehat{X}_n$ is asymptotically optimal.*

Remark 7.

$$\begin{aligned} C = 0 &\Leftrightarrow P(\forall t \in [0,1] : \ \sigma(t, X(t) = 0) = 1 \\ &\Leftrightarrow \text{equation (1) is deterministic} \end{aligned}$$

Can we be as good with

- a fixed number of discretization points?
- a fixed discretization?
- an equidistant discretization?

Interesting subclasses of $\mathbb{X}$

Corresponding to the above questions we consider

$$\begin{aligned}
\mathbb{X}^{\text{fixed } \nu} &= \{\widehat{X}_{(\psi,\chi,\varphi)} \in \mathbb{X} : \nu_{(\psi,\chi,\varphi)} \text{ constant}\} \\
\mathbb{X}^{\text{fixed}} &= \{\widehat{X}_{(\psi,\chi,\varphi)} \in \mathbb{X}^{\text{fixed } \nu} : \psi_\ell \text{ constant for } \ell \le \nu_{(\psi,\chi,\varphi)}\} \\
\mathbb{X}^{\text{equi}} &= \{\widehat{X}_{(\psi,\chi,\varphi)} \in \mathbb{X}^{\text{fixed}} : \psi_\ell = \ell/\nu_{(\psi,\chi,\varphi)} \text{ for } \ell \le \nu_{(\psi,\chi,\varphi)}\}
\end{aligned}$$

with respective minimal errors

$$\begin{aligned}
e_N^{\text{fixed } \nu} &= \inf\{e(\widehat{X}) : \widehat{X} \in \mathbb{X}^{\text{fixed } \nu},\ c(\widehat{X}) \le N\} \\
e_N^{\text{fixed}} &= \inf\{e(\widehat{X}) : \widehat{X} \in \mathbb{X}^{\text{fixed}},\ c(\widehat{X}) \le N\} \\
e_N^{\text{equi}} &= \inf\{e(\widehat{X}) : \widehat{X} \in \mathbb{X}^{\text{equi}},\ c(\widehat{X}) \le N\}
\end{aligned}$$

Note: (Piecewise linear) interpolated Euler schemes and Milstein schemes belong to the class $\mathbb{X}^{\text{fixed}}$.
The (piecewise linear) interpolated equidistant Milstein schemes $\widehat{X}_n^M$ belong to the class $\mathbb{X}^{\text{equi}}$.

Put

$$\begin{aligned}
C^{\text{fixed } \nu} &= \frac{1}{\sqrt{6}} \cdot \left(E\left(\int_0^1 |\sigma(t, X(t))|\, dt \right)^2 \right)^{1/2} \\
C^{\text{fixed}} &= \frac{1}{\sqrt{6}} \cdot \int_0^1 \left(E\big(\sigma^2(t, X(t))\big) \right)^{1/2} dt
\end{aligned}$$

and recall the constant

$$C^{\text{equi}} = \frac{1}{\sqrt{6}} \cdot \left(E \int_0^1 \sigma^2(t, X(t))\, dt \right)^{1/2}$$

Theorem 9. *(Hofmann, M-G, Ritter 2001)*

$$\begin{aligned}
\lim_{N\to\infty} N^{1/2} \cdot e_N^{\textit{fixed } \nu} &= C^{\textit{fixed } \nu} \\
\lim_{N\to\infty} N^{1/2} \cdot e_N^{\textit{fixed}} &= C^{\textit{fixed}} \\
\lim_{N\to\infty} N^{1/2} \cdot e_N^{\textit{equi}} &= C^{\textit{equi}}
\end{aligned}$$

Recall the asymptotic constant for e_N:

$$C = \frac{1}{\sqrt{6}} \cdot E \int_0^1 |\sigma(t, X(t))|\, dt$$

Note that

$$C \le C^{\text{fixed } \nu} \le C^{\text{fixed}} \le C^{\text{equi}}$$

with strict inequality in most cases, see Remark 9.

Example 12. Geometric Brownian motion

$$dX(t) = b \cdot X(t)\, dW(t), \quad X(0) = 1$$

Here

$$C^{\text{equi}} = \frac{1}{\sqrt{6}} \cdot \big(\exp(b^2) - 1\big)^{1/2}$$

$$C^{\text{fixed}} = \frac{1}{\sqrt{6}} \cdot \frac{2}{|b|} \cdot \big(\exp(b^2/2) - 1\big)$$

$$C^{\text{fixed } \nu} = \frac{1}{\sqrt{6}} \cdot \frac{1}{|b|} \cdot \big(2 \cdot \exp(b^2) - b^2 - 1\big)^{1/2}$$

$$C = \frac{1}{\sqrt{6}} \cdot |b|$$

Exponential dependence on b except for the constant C.

Remark 8. (Interpretation of asymptotic constants)

Recall: conditional Hölder constant $|\sigma(t, X(t))|$ describes smoothness of X locally in time and space.

Constants C and $C^{\text{fixed } \nu}$ are based on average

$$H = \int_0^1 |\sigma(t, X(t))|\, dt$$

of the conditional Hölder constant along a trajectory:

$$C = 1/\sqrt{6} \cdot E(H), \quad C^{\text{fixed } \nu} = 1/\sqrt{6} \cdot (E(H^2))^{1/2}$$

Due to (7)

$$(8) \qquad E(X(t+\delta) - X(t))^2 = E|\sigma(t, X(t))|^2 \cdot \delta + o(\delta)$$

Thus

$$\alpha(t) = \big(E|\sigma(t, X(t))|^2\big)^{1/2}$$

describes smoothness of X only locally in time.

Constants C^{fixed} and C^{equi} are based on α:

$$C^{\text{fixed}} = \frac{1}{\sqrt{6}} \cdot \int_0^1 \alpha(t)\, dt, \quad C^{\text{equi}} = \frac{1}{\sqrt{6}} \cdot \left(\int_0^1 \alpha^2(t)\, dt\right)^{1/2}$$

Remark 9. (Equality of asymptotic constants)

In general

$C^{\text{equi}} = C^{\text{fixed}}$ iff α is constant

$C^{\text{fixed}} = C^{\text{fixed } \nu}$

iff $\exists\, t_0 \in [0,1],\ \gamma \in C([0,1])$:

$P\big(\forall t \in [0,1] : |\sigma(t, X(t))| = \gamma(t) \cdot |\sigma(t_0, X(t_0))|\big) = 1$

$C^{\text{fixed } \nu} = C$

iff $P\big(\forall t \in [0,1] : |\sigma(t, X(t))| = \alpha(t)\big) = 1$

For equations with additive noise, i.e., $\sigma^{0,1} = 0$: $C = C^{\text{fixed}}$

Furthermore

$C = C^{\text{fixed } \nu} = C^{\text{fixed}} = C^{\text{equi}} = 0$ iff $C = 0$

iff eq (1) is deterministic

Remark 10. (L_2-approximation and L_2-reconstruction)

Consider weighted Brownian motion

$$Y(t) = \rho(t) \cdot W(t), \quad t \in [0,1],$$

with continuous $\rho : [0,1] \to [0,\infty[$.

Best L_2-approximation of Y based on $Y(t_1), \dots, Y(t_N)$:

$$\widehat{Y}_{(t_1,\dots,t_N)}(t) = E\big(Y(t) \,\big|\, Y(t_1), \dots, Y(t_N)\big)$$

Resulting minimal mean square L_2-error:

$$\widetilde{e}_N = \inf\big\{(E\|Y - \widehat{Y}_{(t_1,\dots,t_N)}\|_2^2)^{1/2} : \, 0 < t_1 < \dots < t_N \le 1\big\}$$

Note: Adaption does not help here since Y is Gaussian

From general results on L_2-reconstruction (see Ritter 2000):

$$\lim_{N\to\infty} N^{1/2} \cdot \widetilde{e}_N = C_\rho := \frac{1}{\sqrt{6}} \cdot \int_0^1 \rho(t)\,dt$$

Taking the weight $\rho = \alpha$ yields

$$C^{\text{fixed}} = C_\alpha$$

Taking the *random weight* $\rho = |\sigma(\cdot, X(\cdot))|$ we get

$$C = E\big(C_{|\sigma(\cdot,X(\cdot))|}\big)$$

Heuristics:

- Locally in time, the solution X behaves like a Brownian motion weighted with α, see (8).

- Locally in time and space, the solution X behaves like a Brownian motion randomly weighted with $|\sigma(\cdot, X(\cdot))|$, see (7).
- Approximation of X at discrete points is *cheap* compared to L_2-approximation of X, see Theorems 4 and 7.

Remark 1. (Asymptotically optimal methods)

For the class $\mathbb{X}$: The adaptive method $\widehat{X}_n$ (Corollary 2)

For the class $\mathbb{X}^{\text{fixed }\nu}$: Same definition as $\widehat{X}_n$ with numbers μ_ℓ (essentially) replaced by

$$\mu_\ell = \left\lceil n \cdot |\sigma(t_\ell, \widehat{X}^M(t_\ell))| \cdot \Big(\sum_{j=0}^{k_n-1} |\sigma(t_j, \widehat{X}^M(t_j))|\Big)^{-1} \right\rceil$$

For the class $\mathbb{X}^{\text{fixed}}$: Piecewise linear interpolation of Milstein scheme with *regular* discretization

$$\int_{t_\ell^{(n)}}^{t_{\ell+1}^{(n)}} \alpha(t)\,dt = \frac{1}{n} \cdot \int_0^1 \alpha(t)\,dt, \quad \ell = 0, 1, \dots, n-1$$

For the class $\mathbb{X}^{\text{equi}}$: Piecewise linear interpolation of the equidistant Milstein scheme $\widehat{X}_n^M$ (Theorems 6 and 8)

Simulation experiments

Equation

$$dX(t) = 3 \cdot X(t)\, dW(t), \quad X(0) = 1.$$

Comparison of fully adaptive method $\widehat{X}_n^*$ with simplified adaptive method $\widehat{X}_n$:

- Error and cost of $\widehat{X}_n^*$ by simulation
- Error and cost of $\widehat{X}_n$ by asymptotic formulas from Theorem 6:

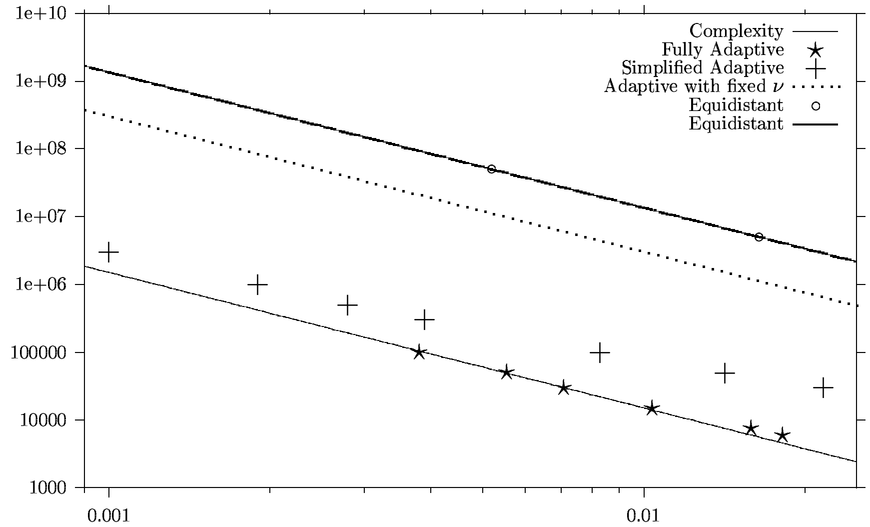
$$c(\widehat{X}_n) \approx 3n, \quad e(\widehat{X}_n) \approx 1/\sqrt{2n}$$

n	**Error**		**Cost**	
	Simul.	Asymp.	Simul.	Asymp.
$2 \cdot 10^3$	$1.82 \cdot 10^{-2}$	$1.58 \cdot 10^{-2}$	6009	6000
$2.5 \cdot 10^3$	$1.59 \cdot 10^{-2}$	$1.41 \cdot 10^{-2}$	7624	7500
$5 \cdot 10^3$	$1.04 \cdot 10^{-2}$	$1.00 \cdot 10^{-2}$	14981	15000
$1 \cdot 10^4$	$7.09 \cdot 10^{-3}$	$7.07 \cdot 10^{-3}$	29843	30000
$1/6 \cdot 10^5$	$5.55 \cdot 10^{-3}$	$5.48 \cdot 10^{-3}$	51297	50000
$1/3 \cdot 10^5$	$3.81 \cdot 10^{-3}$	$3.87 \cdot 10^{-3}$	100122	100000

Conjecture: The fully adaptive method $\widehat{X}_n^*$ is also asymptotically optimal in the class $\mathbb{X}$

Dependence of cost on error within range $[10^{-3}, 2 \cdot 10^{-2}]$:

- Solid or dotted lines by asymptotic formulas from Theorems 6, 7 and 9.
- $\bigstar$ from simulated values in previous table
- $+$ and $\bigcirc$ from explicit formulas for $c(\widehat{X}_n)$ and $e(\widehat{X}_n)$ as well as $e(\widehat{X}_n^M)$



Using the fully or simplified adaptive method the computation time (in terms of the average number of calls to random number generator) decreases by a factor close to

$$\left(C/C^{\text{equi}}\right)^2 = 9/(\exp(9) - 1) = 0.0011108...$$

IV. Optimal approximation at a single point based on point evaluations of W

Classes of methods as in Part III, i.e.,

$\mathbb{X}$: All approximations $\widehat{X}$ that are based on a finite (random) number ν of point evaluations of W

$\mathbb{X}^{\text{fixed }\nu}$: All approximations $\widehat{X} \in \mathbb{X}$ with deterministic ν

$\mathbb{X}^{\text{fixed}}$: All approximations $\widehat{X} \in \mathbb{X}^{\text{fixed }\nu}$ with fixed discretization

$\mathbb{X}^{\text{equi}}$: All approximations $\widehat{X} \in \mathbb{X}^{\text{fixed}}$ with equidistant discretization

Cost measure as in Part III, i.e.,

$$c(\widehat{X}) = E(\nu) = \text{ average number of evaluations of } W$$

Error

$$e(\widehat{X}) = \left(E|X(1) - \widehat{X}(1)|^2\right)^{1/2}$$

Minimal errors

$$e_N = \inf\{e(\widehat{X}) : \widehat{X} \in \mathbb{X},\ c(\widehat{X}) \le N\}$$

Canonical definition of minimal errors $e_N^{\text{fixed }\nu}$, e_N^{fixed}, e_N^{equi}

Assumption throughout the rest of Part IV:

$$a^{(i,j)},\ \sigma^{(i,j)}, \quad i = 0,1,2,\ j = 0,1,2,3,$$

exist and are continuous and bounded.

Example 13. (Geometric Brownian motion)

$$dX(t) = \mu \cdot X(t)\,dt + b \cdot X(t)\,dW(t), \quad X(0) = 1$$

Here

$$X(1) = \exp\bigl((\mu - b^2/2) + b \cdot W(1)\bigr), \quad e_N = e_1 = 0$$

Example 14. (Additive noise)

$$dX(t) = a(t)\,dt + \sigma(t)\,dW(t), \quad X(0) = 0,$$

Here

$$X(1) = \int_0^1 a(t)\,dt + \sigma(1) \cdot W(1) - \int_0^1 \sigma'(t) \cdot W(t)\,dt$$

Essentially, approximation of

$$\int_0^1 \sigma'(t) \cdot W(t)\,dt$$

Example 15. (Langevin equation)

$$dX(t) = \alpha \cdot X(t)\,dt + dW(t), \quad X(0) = 1$$

Here

$$X(1) = W(1) + \int_0^1 \alpha \cdot e^{\alpha\cdot(1-t)} \cdot W(t)\,dt$$

Essentially, approximation of

$$\int_0^1 \alpha \cdot e^{\alpha\cdot(1-t)} \cdot W(t)\,dt$$

Thus: Mean square approximation of X at a single point is strongly connected to approximation of the integral of a weighted Brownian motion

The weighting process

Define

$$\Theta(t) = \eta(t, X(t)) \cdot \Phi(t), \quad t \in [0,1],$$

where

$$\eta = \sigma a^{(0,1)} - \sigma^{(1,0)} - a\sigma^{(0,1)} - 1/2 \cdot \sigma^2 \sigma^{(0,2)}$$

and

$$\Phi(t) = \exp\Big(\int_t^1 \big(a^{(0,1)} - 1/2 \cdot (\sigma^{(0,1)})^2\big)(u, X(u))\,du + \int_t^1 \sigma^{(0,1)}(u, X(u))\,dW(u) \Big)$$

Motivation for η:

General results on approximation of the integral of a weighted Brownian motion suggest (for nonzero weight)

$$(*) \qquad e_N \asymp 1/N,$$

see Ritter 2000. (Upper bound is a consequence of Theorem 4)

For the Wagner-Platen scheme $\widehat{X}_N^{3/2}$ with N equidistant discretization points

$$\big(E|X(1) - \widehat{X}_N^{3/2}(1)|^2\big)^{1/2} \le c/N^{3/2},$$

see Theorem 5.

Note that $\widehat{X}_N^{3/2} \notin \mathbb{X}$ since this method uses integrals of the Brownian motion, see Part II.

If $(*)$ is true then approximation of $X(1)$ is equivalent to approximation of $\widehat{X}_N^{3/2}(1)$.

The function η is composed from the coefficient function associated with the Brownian integrals used by the Wagner-Platen scheme

$$\eta = f_{(1,0)} - f_{(0,1)}$$

$$I_{(1,0),t_\ell,t_{\ell+1}} = \int_{t_\ell}^{t_{\ell+1}} (W(t) - W(t_\ell))\,dt$$

$$I_{(0,1),t_\ell,t_{\ell+1}} = (t_{\ell+1} - t_\ell) \cdot ((W(t_{\ell+1}) - W(t_\ell)) - \int_{t_\ell}^{t_{\ell+1}} (W(t) - W(t_\ell))\,dt$$

All other Itô-integrals are functions of point evaluations of W

Interpretation of the process Φ:

Roughly, $\Phi(t)$ is the mean square derivative of the solution at the final time point $X(1)$ with respect to its state at time t.

For $x \in \mathbb{R}$ and $t \in [0,1]$ consider the equation

$$dX_{t,x}(s) = a(s, X_{t,x}(s))\, ds + \sigma(s, X_{t,x}(s))\, dW(s), \quad t \le s \le 1,$$

with initial value $X_{t,x}(t) = x$.

Note that

$$P^{(X_{t,x}(s))_{t\le s\le 1}} = P^{(X(s))_{t\le s\le 1}\,|\,X(t)=x}$$

For every $s \in [t,1]$ there exists the mean square derivative $X'_{t,x}(s)$ of $X_{t,x}(s)$ with respect to the initial value x, i.e.,

$$\lim_{h\to 0} E\big(1/h \cdot (X_{t,x+h}(s) - X_{t,x}(s)) - X'_{t,x}(s)\big)^2 = 0.$$

The derivative process is explicitly given by

$$X'_{t,x}(s) = \exp\bigg(\int_t^s \big(a^{(0,1)} - 1/2 \cdot (\sigma^{(0,1)})^2\big)(u, X_{t,x}(u))\, du + \int_t^s \sigma^{(0,1)}(u, X_{t,x}(u))\, dW(u)\bigg).$$

Taking $s = 1$ and replacing $X_{t,x}$ by the solution X in the right hand side above yields the definition of $\Phi(t)$.

Analysis of minimal errors

Define the constants

$$C = \frac{1}{\sqrt{12}} \cdot \left(E \int_0^1 |\Theta(t)|^{2/3}\, dt\right)^{3/2}$$

$$C^{\text{fixed }\nu} = \frac{1}{\sqrt{12}} \cdot \left(E\left(\int_0^1 |\Theta(t)|^{2/3}\, dt\right)^3\right)^{1/2}$$

$$C^{\text{fixed}} = \frac{1}{\sqrt{12}} \cdot \left(\int_0^1 \big(E|\Theta(t)|^2\big)^{1/3}\, dt\right)^{3/2}$$

$$C^{\text{equi}} = \frac{1}{\sqrt{12}} \cdot \left(\int_0^1 E|\Theta(t)|^2\, dt\right)^{1/2}$$

Theorem 10. *(M-G 2004)*

$$\lim_{N\to\infty} N \cdot e_N = C$$

$$\lim_{N\to\infty} N \cdot e_N^{\textit{fixed }\nu} = C^{\textit{fixed }\nu}$$

$$\lim_{N\to\infty} N \cdot e_N^{\textit{fixed}} = C^{\textit{fixed}}$$

$$\lim_{N\to\infty} N \cdot e_N^{\textit{equi}} = C^{\textit{equi}}$$

Note: Constants are alltogether either zero or positve
In the latter case the order of convergence is $1/N$ for all classes, however

$$C \le C^{\text{fixed }\nu} \le C^{\text{fixed}} \le C^{\text{equi}}$$

with strict inequality in most cases, see Remark 13

Example 16. (Additive noise)

$$dX(t) = a(t)\,dt + \sigma(t)\,dW(t), \quad X(0) = 0,$$

Here

$$\eta = -\sigma', \quad \Phi(t) \equiv 1$$

Thus deterministic weighting process

$$\Theta = \eta = -\sigma'$$

yields

$$C = C^{\text{fixed }\nu} = C^{\text{fixed}} = \frac{1}{\sqrt{12}} \cdot \Big(\int_0^1 |\sigma'(t)|^{2/3}\,dt\Big)^{3/2}$$

$$C^{\text{equi}} = \frac{1}{\sqrt{12}} \cdot \Big(\int_0^1 |\sigma'(t)|^2\,dt\Big)^{1/2}$$

similar to mean square L_2-approximation

Example 17. (Linear equation)

$$dX(t) = \alpha(t) \cdot X(t)\,dt + \beta(t) \cdot X(t)\,dW(t), \quad X(0) = 1,$$

Solution

$$X(t) = \exp\Big(\int_0^t (\alpha - 1/2 \cdot \beta^2)(u)\ du + \int_0^t \beta(u)\ dW(u)\Big)$$

Here

$$\eta(t, x) = -\beta'(t) \cdot x$$

and

$$\Phi(t) = X(1)/X(t)$$

Weighting process

$$\Theta(t) = -\beta'(t) \cdot X(1)$$

yields constants

$$C = \frac{1}{\sqrt{12}} \cdot \|\beta'\|_{2/3} \cdot \exp\big(\|\alpha\|_1 - 1/6 \cdot \|\beta\|_2^2\big)$$

$$C^{\text{fixed }\nu} = C^{\text{fixed}} = \frac{1}{\sqrt{12}} \cdot \|\beta'\|_{2/3} \cdot \exp\big(\|\alpha\|_1 + 1/2 \cdot \|\beta\|_2^2\big)$$

$$C^{\text{equi}} = \frac{1}{\sqrt{12}} \cdot \|\beta'\|_2 \cdot \exp\big(\|\alpha\|_1 + 1/2 \cdot \|\beta\|_2^2\big)$$

For instance, if $\alpha = 0$ and $\beta(t) = b \cdot t$ then

$$C = |b|/\sqrt{12} \cdot \exp(-b^2/18)$$

$$C^{\text{fixed }\nu} = C^{\text{fixed}} = C^{\text{equi}} = |b|/\sqrt{12} \cdot \exp(b^2/6)$$

C is exponentially decreasing in b while the other constants are exponentially increasing in b.

Adapting the number ν of evaluations of W to the trajectory of the solution X is essential here.

Remark 12.

Theorem 10 determines rate of convergence of minimal errors only for nonzero constants.

Clearly: $\quad C = C^{\text{fixed }\nu} = C^{\text{fixed}} = C^{\text{equi}} = 0 \quad$ iff

(9) $$P(\forall t \in [0,1] :\ \eta(t, X(t)) = 0) = 1$$

Conjecture:

$$(9) \quad \text{iff} \quad \exists \text{ measurable } f : [0,1] \times \mathbb{R} \to \mathbb{R} : \quad P(\forall t \in [0,1] : X(t) = f(t, W(t)) = 1$$

If the conjecture is true then

$$e_N \asymp 1/N \quad \text{or} \quad e_N = e_1 = 0.$$

The conjecture is known to be true in many cases:

- $\inf_{t,x} |\sigma(t,x)| > 0$
  (Clark and Cameron 1980)
- a and σ have parial derivatives of any order
  (Yamoto 1979)
- For the linear equation from Example 17
  ((9) implies $\beta' = 0$)

Remark 13. (Equality of asymptotic constants)

$$C^{\text{equi}} = C^{\text{fixed}}$$
$$\text{iff} \quad \exists\, \theta \in \mathbb{R}\ \forall t \in [0,1] : E(\Theta^2(t) = \theta$$

$$C^{\text{fixed}} = C^{\text{fixed}\ \nu}$$
$$\text{iff} \quad \exists\, t_0 \in [0,1],\ \theta \in C([0,1]) : \quad P\big(\forall t \in [0,1] : \Theta(t) = \theta(t) \cdot \Theta(t_0)\big) = 1$$

$$C^{\text{fixed}\ \nu} = C$$
$$\text{iff} \quad \exists\, \theta \in C([0,1]) : \ P\big(\forall t \in [0,1] : \Theta(t) = \theta(t)\big) = 1$$

See Examples 16 and 17

Remark 14. (mean square approximation at a single point and weighted integration)

Consider integral of weighted Brownian motion

$$Y = \int_0^1 \rho(t) \cdot W(t)$$

with continuous $\rho : [0,1] \to [0,\infty[$.

Best mean square approximation of Y based on $W(t_1), \ldots, W(t_N)$:

$$\widehat{Y}_{(t_1,\ldots,t_N)} = E\big(Y \,\big|\, W(t_1), \ldots, W(t_N)\big)$$

Resulting minimal mean square error:

$$\widetilde{e}_N = \inf\Big\{ \big(E\big|Y - \widehat{Y}_{(t_1,\ldots,t_N)}\big|^2\big)^{1/2} : \ 0 < t_1 < \ldots < t_N \le 1 \Big\}$$

Note: Adaption does not help here since Y is Gaussian

From general results on weighted integration (see Ritter 2000):

$$\lim_{N\to\infty} N \cdot \widetilde{e}_N = C_\rho := \frac{1}{\sqrt{12}} \cdot \left(\int_0^1 |\rho(t)|^{2/3}\, dt \right)^{3/2}$$

We have

$$C^{\text{fixed}} = C_{(E|\Theta(\cdot)|^2)^{1/2}}$$

Taking the *random weight* $\rho = \Theta$ yields

$$C = \Big(E\big(C_\Theta^{2/3}\big)\Big)^{3/2}$$

The adaptive approximation $\widehat{X}_n(1)$

Basic idea:

- Replace integrals
$$J_\ell = \int_{t_\ell}^{t_{\ell+1}} \big(W(t) - W(t_\ell)\big)\, dt$$
in the definition of the Wagner-Platen scheme by suitable approximations based on point evaluations of W.
- Adapt the discretization to the random weight Θ:

  The *larger* the value of $\Theta(t)$
  the *finer* the discretization around t

Algorithm (essential structure):

1) Dependent on n determine *coarse* equidistant prediscretization
$$t_\ell = \ell/k_n, \quad \ell = 0, 1, \ldots, k_n$$
2) Compute corresponding *truncated* Wagner-Platen scheme
$$\widehat{X}^{\text{trunc}}(t_\ell), \quad \ell = 0, 1, \ldots, k_n$$
3) Use *truncated* Wagner-Platen scheme to compute a discrete approximation of the random weight
$$\widehat{\Theta}(t_\ell), \quad \ell = 0, 1, \ldots, k_n - 1$$

4) Determine number of additional equidistant points in subintervals $]t_\ell, t_{\ell+1}[$ by
$$\mu_\ell = \left\lceil n/k_n \cdot |\widehat{\Theta}(t_\ell)|^{2/3} \right\rceil$$
Resulting points in $[t_\ell, t_{\ell+1}]$
$$\tau_{\ell,j} = t_\ell + j \cdot (k_n \cdot (\mu_\ell + 1))^{-1}, \quad j = 0, 1, \ldots, \mu_\ell + 1$$
5) Estimate integrals J_ℓ by (adaptive) trapezoidal rule
$$\widehat{J}_\ell = \frac{1}{2k_n \cdot (\mu_\ell + 1)} \cdot \sum_{j=0}^{\mu_\ell} \big(W(\tau_{\ell,j}) + W(\tau_{\ell,j+1}) - 2W(t_\ell)\big)$$
6) Compute approximation at $t = 1$
$$\widehat{X}_n(1) = \widehat{X}^{\text{trunc}}(1) + \sum_{\ell=0}^{k_n-1} \widehat{\Theta}(t_\ell) \cdot \widehat{J}_\ell$$

Note that $\widehat{X}_n(1) \in \mathbb{X}$.

Missing details:

ad 1) Choose k_n such that
$$\lim_{n\to\infty} k_n/n = 0$$
(keeping track of size of random weight)

and
$$\lim_{n\to\infty} k_n^{3/2}/n = \infty$$
(estimating size of random weight sufficiently well)

ad 2) *Truncated* Wagner-Platen scheme

$$
\begin{aligned}
&\widehat{X}^{\text{trunc}}(t_0) = x_0,\\
&\widehat{X}^{\text{trunc}}(t_{\ell+1})\\
&\quad = \widehat{X}^{\text{trunc}}(t_\ell) + a(t_\ell, \widehat{X}^{\text{trunc}}(t_\ell)) \cdot (t_{\ell+1} - t_\ell)\\
&\qquad\qquad + \sigma(t_\ell, \widehat{X}^{\text{trunc}}(t_\ell)) \cdot (W(t_{\ell+1}) - W(t_\ell))\\
&\qquad + 1/2 \cdot \big(\sigma^{(0,1)} \cdot \sigma\big)(t_\ell, \widehat{X}^{\text{trunc}}(t_\ell))\\
&\qquad\qquad \times \big((W(t_{\ell+1}) - W(t_\ell))^2 - (t_{\ell+1} - t_\ell)\big)\\
&\qquad + \big(\sigma^{(1,0)} + a \cdot \sigma^{(0,1)} - \sigma/2 \cdot (\sigma^{(0,1)})^2\big)(t_\ell, \widehat{X}^{\text{trunc}}(t_\ell))\\
&\qquad\qquad \times \big(W(t_{\ell+1}) - W(t_\ell)\big) \cdot (t_{\ell+1} - t_\ell)\\
&\qquad + 1/6 \cdot \big(\sigma \cdot (\sigma^{(0,1)})^2 + \sigma^2 \cdot \sigma^{(0,2)}\big)(t_\ell, \widehat{X}^{\text{trunc}}(t_\ell))\\
&\qquad\qquad \times \big(W(t_{\ell+1}) - W(t_\ell)\big)^3\\
&\qquad + 1/2 \cdot \big(a^{(1,0)} + a \cdot a^{(0,1)} + \sigma^2/2 \cdot a^{(0,2)}\big)(t_\ell, \widehat{X}^{\text{trunc}}(t_\ell))\\
&\qquad\qquad \times (t_{\ell+1} - t_\ell)^2\\
&\quad = \widehat{X}^{\text{trunc}}(t_\ell) + \sum_{\alpha \in A_{3/2} \setminus \{(1,0),(0,1)\}} f_\alpha\big(t_\ell, \widehat{X}^{\text{trunc}}(t_\ell)\big) \cdot I_{\alpha, t_\ell, t_{\ell+1}}\\
&\qquad + f_{(0,1)}\big(t_\ell, \widehat{X}^{\text{trunc}}(t_\ell)\big) \cdot (t_{\ell+1} - t_\ell) \cdot (W(t_{\ell+1}) - W(t_\ell))
\end{aligned}
$$

Note: This scheme uses only point evaluations of W

ad 3) Estimation of random weight

Recall

$$\Theta(t) = \eta(t, X(t)) \cdot \Phi(t)$$

Consider random field

$$\Psi(t, s) = \Phi(t)/\Phi(s), \quad t \le s \le 1$$

Note: $\Phi(t) = \Psi(t, 1)$ and $\Psi(t, \cdot)$ satisfies

$$
\begin{aligned}
d\Psi(t, s) &= a^{(0,1)}(s, X(s)) \cdot \Psi(t, s)\, ds\\
&\quad + \sigma^{(0,1)}(s, X(s)) \cdot \Psi(t, s)\, dW(s), \quad t \le s \le 1,
\end{aligned}
$$

Euler-type scheme for approximation of $\Psi(t_\ell, \cdot)$

$$
\begin{aligned}
&\widehat{\Psi}(t_\ell, t_\ell) = 1,\\
&\widehat{\Psi}(t_\ell, t_{j+1}) = \widehat{\Psi}(t_\ell, t_j)\\
&\qquad + a^{(0,1)}(t_j, \widehat{X}^{\text{trunc}}(t_j)) \cdot \widehat{\Psi}(t_\ell, t_j) \cdot (t_{j+1} - t_j)\\
&\qquad + \sigma^{(0,1)}(t_j, \widehat{X}^{\text{trunc}}(t_j)) \cdot \widehat{\Psi}(t_\ell, t_j) \cdot (W(t_{j+1}) - W(t_j))
\end{aligned}
$$

for $j = \ell, \ell + 1, \ldots, k_n - 1$ yields approximation of $\Phi(t_\ell)$

$$\widehat{\Phi}(t_\ell) = \widehat{\Psi}(t_\ell, 1)$$

Use

$$\widehat{\Theta}(t_\ell) = \eta(t_\ell, \widehat{X}^{\text{trunc}}(t_\ell)) \cdot \widehat{\Phi}(t_\ell), \quad \ell = 0, 1, \ldots, k_n - 1.$$

Recall from Theorem 10

$$\lim_{n\to\infty} N \cdot e_N = C$$

with

$$C = \frac{1}{\sqrt{12}} \cdot \left(E \int_0^1 |\Theta(t)|^{2/3} \, dt \right)^{3/2}$$

Theorem 1. *(M-G 2004)*

(i) $$\lim_{n\to\infty} c\big(\widehat{X}_n\big) \cdot e\big(\widehat{X}_n\big) = C,$$

(ii) $$\lim_{n\to\infty} n^{-1} \cdot c\big(\widehat{X}_n\big) = \sqrt{12} \cdot C.$$

Corollary 3. *If $C > 0$ then $\widehat{X}_n(1)$ is asymptotically optimal in the class $\mathbb{X}$ for mean square approximation of $X(1)$.*

Remark 15. (Asymptotically optimal methods for subclasses)

Same definition as $\widehat{X}_n(1)$ with numbers μ_ℓ (essentially) replaced by

$$\mu_\ell = \left\lceil n \cdot |\widehat{\Theta}(t_\ell)|^{2/3} \cdot \Big(\sum_{j=0}^{k_n-1} |\widehat{\Theta}(t_j)|^{2/3} \Big)^{-1} \right\rceil \qquad \text{for } \mathbb{X}^{\text{fixed } \nu},$$

$$\mu_\ell = \left\lceil n \cdot \big(E|\Theta(t_\ell)|^2\big)^{1/3} \cdot \Big(\sum_{j=0}^{k_n-1} \big(E|\Theta(t_j)|^2\big)^{1/3} \Big)^{-1} \right\rceil \qquad \text{for } \mathbb{X}^{\text{fixed}},$$

and $\mu_\ell = 0$ for the class $\mathbb{X}^{\text{equi}}$.

Remark 16. (Performance of Milstein scheme)

The Milstein scheme is, in general, not asymptotically optimal for approximation of the solution at a single point.

Example:

$$dX(t) = \sigma(t)\, dW(t), \quad X(0) = 0$$

Milstein scheme based on fixed discretization

$$0 = t_0 < t_1 < \ldots < t_n = 1$$

is given by

$$\widehat{X}^M_{(t_1,\ldots,t_n)}(t_\ell) = \sum_{j=0}^{\ell-1} \sigma(t_j) \cdot (W(t_{j+1}) - W(t_j))$$

and satisfies

$$e\big(\widehat{X}^M_{(t_1,\ldots,t_n)}\big) = \frac{1}{3} \sum_{\ell=0}^{n-1} \big(\sigma'(t_\ell)\big)^2 \cdot (t_{\ell+1} - t_\ell)^3.$$

It follows

$$\lim_{n\to\infty} n \cdot \inf_{0<t_1<\cdots<t_n=1} e\big(\widehat{X}^M_{(t_1,\ldots,t_n)}\big) = 2 \cdot C^{\text{fixed}}$$

and

$$\lim_{n\to\infty} n \cdot e\big(\widehat{X}^M_{(1/n,2/n,\ldots,1)}\big) = 2 \cdot C^{\text{equi}}$$

V. Outlook

	Treated in lecture	What about?
global error	$\left(E\|X-\widehat{X}\|_2^2\right)^{1/2}$	$\left(E\|X-\widehat{X}\|_p^q\right)^{1/q}$
error at $t=1$	$\left(E\lvert X(1)-\widehat{X}(1)\rvert^2\right)^{1/2}$	$\left(E\lvert X(1)-\widehat{X}(1)\rvert^q\right)^{1/q}$
information about W	point evaluations	cont. lin. functionals iterated Itô-integrals
dimension	scalar equations	systems

Optimal global approximation based on

- point evaluations of W:

  Hofmann, M-G, Ritter 2001: $p=q=2$, scalar equations

  M-G 2002a: $p<\infty$, $q\geq 1$, systems

  M-G 2002b: $p=\infty$, $q\geq 1$, systems
- continuous linear functionals of W:

  Hofmann, M-G, Ritter 2002: $p=q=2$, scalar equations
- iterated Itô-integrals:

  Hofmann, M-G 2004: $p=q=2$, scalar equations

L_p-approximation of scalar equations:

Define

$$C_{p,q}(a,\sigma,x_0)=\left(E\left\|\sigma(\cdot,X(\cdot))\right\|_{2p/(p+2)}^{2q/(q+2)}\right)^{(q+2)/2q},$$

for $q\in[1,\infty[$, $p\in[1,\infty]$ $(2\infty/(\infty+2):=2)$, and put

$$m_p=\left(\int_{-\infty}^{\infty}|z|^p\cdot(2\pi)^{-1/2}\cdot\exp(-z^2/2)\,dz\right)^{1/p}$$

for $p\in[1,\infty[$.

info about W	min. errors order	asymp. const.	asymp. opt. meth. essentially
point evaluation $1\leq p=q<\infty$	$N^{-1/2}$	$\frac{m_p}{\sqrt{6}}\cdot C_{p,q}(a,\sigma,x_0)$	Milstein with stepsize $\sim\lvert\sigma(t,X(t))\rvert^{-2p/(p+2)}$
$p=\infty$ $1\leq q<\infty$	$(N/\ln N)^{-1/2}$	$\frac{1}{\sqrt{2}}\cdot C_{\infty,q}(a,\sigma,x_0)$	Euler with stepsize $\sim\lvert\sigma(t,X(t))\rvert^{-2}$
cont. lin. func. $p=q=2$	$N^{-1/2}$	$\frac{1}{\pi}\cdot C_{2,2}(a,\sigma,x_0)$	Uses adaptive Karhunen-Loeve expansion of W
iter. Itô-integrals $p=q=2$	$N^{-1/2}$ (non-adaptive)	?	?

General picture seems to be:

- order of convergence of minimal errors

$$= \begin{cases} N^{-1/2} & \text{if } p < \infty \\ (N/\ln N)^{-1/2} & \text{if } p = \infty \end{cases}$$

- asymptotic constant

$$= \kappa_{p,q}(\text{info}) \cdot C_{p,q}(a, \sigma, x_0)$$

  where $\kappa_{p,q}(\text{info})$ only depends on error parameters p, q, and the type of information about W.

- key quantity for asymptotic constants and for construction of asymptotically optimal methods

$$= \text{ conditional Hölder constant } |\sigma(t, X(t))|$$

L_p-approximation of systems based on point evaluations of W:

- order of convergence of minimal errors as above
- key quantities for asymptotic constants and for construction of asymptotically optimal methods:

  Conditional Hölder constants of components of solution
  and
  Degree of non-commutativity of diffusion coefficient

Optimal approximation at $t = 1$ based on

- point evaluations of W:

  M-G 2004: $q \geq 1$, scalar equations
  M-G 2002a: $q = 2$, non-commutative systems

- continuous linear functionals of W or iterated Itô-integrals:

  Only upper bounds from Theorem 5, see Part II.

Scalar equations and point evaluations of W:

- order of convergence of minimal errors

$$= N^{-1} \quad \text{(up to specific cases)}$$

- key quantity for asymptotic constants and for construction of asymptotically optimal methods:

  random weight composed of Itô-Taylor coefficient functions and means square derivative of solution

Non-commutative systems and point evaluations of W:

- order of convergence of minimal errors

$$= N^{-1/2}$$

- key quantity for asymptotic constants and for construction of asymptotically optimal methods:

  random weight composed of degree of non-commutativity and means square derivative of solution

Appendix A. Proof of Theorem 4(*i*)

Define a continuous version of the Milstein scheme by

$$\begin{aligned}\widetilde{X}^M(t_0) &= x_0,\\ \widetilde{X}^M(t) &= \widehat{X}^M(t_\ell) + a(t_\ell, \widehat{X}^M(t_\ell)) \cdot (t - t_\ell)\\ &\quad + \sigma(t_\ell, \widehat{X}^M(t_\ell)) \cdot (W(t) - W(t_\ell))\\ &\quad + 1/2 \cdot \big(\sigma^{(0,1)} \cdot \sigma\big)(t_\ell, \widehat{X}^M(t_\ell))\\ &\qquad \times \big((W(t) - W(t_\ell))^2 - (t - t_\ell)\big).\end{aligned}$$

for $t \in [t_\ell, t_{\ell+1}]$, $\ell = 0, 1, \ldots, n-1$.

Note: $\quad \widetilde{X}^M(t_\ell) = \widehat{X}^M(t_\ell)$

Remark 17. The process $\widetilde{X}^M$ is not a numerical method for global approximation of X since the complete trajectories of W are needed for its construction.

Notation $\quad |Y|_q := (E|Y|^q)^{1/q}$ for a random variable Y

Instead of Theorem 4(i) we prove the stronger

Theorem 12. *Under the assumptions of Theorem 4*

$$\sup_{t\in[0,1]} |X(t) - \widetilde{X}^M(t)|_q \le c \cdot \Delta_{\max}.$$

The main tool for the proof is

Lemma 1. *(Gronwall's inequality)*
Let $\alpha, \beta \in \mathbb{R}$ with $\beta \ge 0$, and $f : [0,1] \to [0,\infty[$ be bounded and Borel-measurable. If

$$\forall t \in [0,1]: \qquad f(t) \le \alpha + \beta \cdot \int_0^t f(s)\, ds$$

then

$$\sup_{t\in[0,1]} f(t) \le \alpha \cdot \exp(\beta).$$

For a proof see, e.g., Revuz and Yor 1991.

We start with moment inequalities for $\widetilde{X}^M$.

Lemma 2. *Under the assumptions of Theorem 4*

(i) $$\sup_{t\in[0,1]} |\widetilde{X}^M(t)|_q \le c,$$

(ii) $$\sup_{t\in[0,1]} |\widetilde{X}^M(s) - \widetilde{X}^M(t)|_q \le c \cdot |s-t|^{1/2},$$

Proof. Define $f : [0,1] \to [0,\infty]$ by

$$f(t) = \sup_{s\in[0,t]} |\widetilde{X}^M(s)|_q .$$

Clearly, f is Borel-measurable. We show that f is bounded. Let $t \in [t_\ell, t_{\ell+1}]$ and observe the regularity conditions on a and σ to obtain

$$
\begin{aligned}
(10)\quad & |\widetilde{X}^M(t) - \widetilde{X}^M(t_\ell)|_q \\
&\le |a(t_\ell, \widetilde{X}^M(t_\ell)) \cdot (t - t_\ell)|_q \\
&\qquad + |\sigma(t_\ell, \widetilde{X}^M(t_\ell)) \cdot (W(t) - W(t_\ell))|_q \\
&\qquad + \big|\big(\sigma \cdot \sigma^{(0,1)}\big)(t_\ell, \widetilde{X}^M(t_\ell)) \\
&\qquad\qquad \times \big((W(t) - W(t_\ell))^2 - (t - t_\ell)\big)\big|_q \\
&\le (t - t_\ell) \cdot |a(t_\ell, \widetilde{X}^M(t_\ell))|_q \\
&\qquad + c \cdot (t - t_\ell)^{1/2} \cdot |\sigma(t_\ell, \widetilde{X}^M(t_\ell))|_q \\
&\le c \cdot (t - t_\ell)^{1/2} \cdot \big(1 + |\widetilde{X}^M(t_\ell)|_q\big),
\end{aligned}
$$

which yields $f(1) < \infty$ since $|\widetilde{X}^M(t_0)|_q = |x_0|$.

Next, we use Gronwall's inequality to prove an upper bound for f, which does not depend on the discretization. Put

$$
Z(s) = \sum_j \sigma(t_j, \widetilde{X}^M(t_j)) \cdot 1_{]t_j, t_{j+1}]}(s)
$$

and note that

$$
|Z(s)|_q \le c \cdot \big(1 + \sup_{u \in [0,s]} |\widetilde{X}^M(u)|_q\big) \le c \cdot (1 + f(s)).
$$

Assume $q \ge 2$. By the Burkholder martingale moment inequality,

$$
\begin{aligned}
|\widetilde{X}^M(t)|_q &\le |x_0| + c \cdot \sum_{j=0}^{\ell-1} \big(1 + |\widetilde{X}^M(t_j)|_q\big) \cdot (t_{j+1} - t_j) \\
&\qquad + c \cdot \big(1 + |\widetilde{X}^M(t_\ell)|_q\big) \cdot (t - t_\ell) + \Big| \int_0^t Z(s)\, dW(s) \Big|_q
\end{aligned}
$$

$$
\begin{aligned}
&\le c \cdot \Big(1 + \int_0^t f(s)\, ds\Big) + c \cdot \Big(\int_0^t |Z(s)|_q^q\, ds\Big)^{1/q} \\
&\le c \cdot \Big(1 + \int_0^t f(s)\, ds + \Big(\int_0^t (f(s))^q\, ds\Big)^{1/q}\Big) \\
&\le c \cdot \Big(1 + \int_0^t (f(s))^q\, ds\Big)^{1/q}.
\end{aligned}
$$

We conclude that

$$
f^q(t) \le c + c \cdot \int_0^t f^q(s)\, ds
$$

for all $t \in [0,1]$, which implies inequality (i) by Gronwall's Lemma. Clearly, inequality (ii) is a consequence of (10) and inequality (i). $\square$

Next, let

$$
V(t) = \sum_\ell \big(a^{(0,1)} \cdot \sigma\big)(t_\ell, \widetilde{X}^M(t_\ell)) \cdot (W(t) - W(t_\ell)) \cdot 1_{]t_j, t_{j+1}]}(t)
$$

and define

$$
U(t) = \int_0^t V(s)\, ds.
$$

Lemma 3. *Under the assumptions of Theorem 4*

$$
\sup_{t \in [0,1]} |U(t)|_q \le c \cdot \Delta_{\max}.
$$

Proof. Assume $q \in N$ and let $(\mathcal{F}_t)_{0\le t\le 1}$ denote the filtration that is generated by W. Let $t \in [t_\ell, t_{\ell+1}]$. By the regularity properties of a and σ,

$$\begin{aligned}
E\big(|U(t)|^{2q}\big|\mathcal{F}_{t_\ell}\big)
&= E\bigg(\bigg(U(t_\ell) + \int_{t_\ell}^t V(s)\,ds\bigg)^{2q}\bigg|\mathcal{F}_{t_\ell}\bigg) \\
&= \sum_{r=0}^{2q}\binom{2q}{r}\cdot U^{2q-r}(t_\ell)\cdot E\bigg(\bigg(\int_{t_\ell}^t V(s)\,ds\bigg)^r\bigg|\mathcal{F}_{t_\ell}\bigg) \\
&= \sum_{r=0}^{2q}\binom{2q}{r}\cdot U^{2q-r}(t_\ell)\cdot \big(a^{(0,1)}\cdot\sigma\big)^r(t_\ell,\widetilde{X}^M(t_\ell)) \\
&\qquad \times E\bigg(\int_{t_\ell}^t (W(s)-W(t_\ell))\,ds\bigg)^r \\
&\le U^{2q}(t_\ell) + c\cdot\sum_{r=1}^{q}\binom{2q}{2r}\cdot U^{2q-2r}(t_\ell) \\
&\qquad \times \big(a^{(0,1)}\cdot\sigma\big)^{2r}(t_\ell,\widetilde{X}^M(t_\ell))\cdot(t-t_\ell)^{3r} \\
&\le U^{2q}(t_\ell) + c\cdot(t-t_\ell)\cdot\sum_{r=1}^{q}\binom{2q}{2r}\cdot U^{2q-2r}(t_\ell) \\
&\qquad \times \big((t-t_\ell)\cdot(1+|\widetilde{X}^M(t_\ell)|)\big)^{2r}
\end{aligned}$$

Lemma 2 yields

$$\begin{aligned}
E\big(U^{2q-2r}(t_\ell)\cdot(1+|\widetilde{X}^M(t_\ell)|)^{2r}\big)
&\le |U(t_\ell)|_{2q}^{2q-2r}\cdot\big|1+|\widetilde{X}^M(t_\ell)|\big|_{2q}^{2r} \\
&\le c\cdot|U(t_\ell)|_{2q}^{2q-2r}.
\end{aligned}$$

Hence

$$\begin{aligned}
&|U(t)|_{2q}^{2q} \\
&\quad\le |U(t_\ell)|_{2q}^{2q} + c\cdot(t-t_\ell)\cdot\sum_{r=1}^{q}\binom{2q}{2r}\cdot|U(t_\ell)|_{2q}^{2q-2r}\cdot(t-t_\ell)^{2r} \\
&\quad\le |U(t_\ell)|_{2q}^{2q} + c\cdot(t-t_\ell)\cdot\big(|U(t_\ell)|_{2q}+(t-t_\ell)\big)^{2q} \\
&\quad\le |U(t_\ell)|_{2q}^{2q}\cdot(1+c\cdot(t-t_\ell)) + c\cdot(t-t_\ell)^{2q+1},
\end{aligned}$$

which implies

$$\sup_{s\in[0,t]}|U(s)|_{2q}^{2q} \le c\cdot\Delta_{\max}^{2q} + c\cdot\int_0^t \sup_{s\in[0,u]}|U(s)|_{2q}^{2q}\,du$$

for every $t\in[0,1]$. Use Gronwall's Lemma to complete the proof. □

Proof of Theorem 12.

By definition, $X(t) = x_0 + A(t) + \Sigma(t)$, where

$$A(t) = \int_0^t a(s,X(s))\,ds, \quad \Sigma(t) = \int_0^t \sigma(s,X(s))\,dW(s).$$

Similarly, $\widetilde{X}^M(t) = x_0 + \widetilde{A}(t) + \widetilde{\Sigma}(t)$, where

$$\widetilde{A}(t) = \int_0^t \sum_{\ell=0}^{n-1} a(t_\ell,\widetilde{X}^M(t_\ell))\cdot 1_{]t_\ell,t_{\ell+1}]}(s)\,ds$$

$$\widetilde{\Sigma}(t) = \int_0^t \sum_{\ell=0}^{n-1} \sigma(t_\ell, \widetilde{X}^M(t_\ell)) \cdot 1_{]t_\ell, t_{\ell+1}]}(s)\, dW(s)$$
$$+ \int_0^t \sum_{\ell=0}^{n-1} \big(\sigma \cdot \sigma^{(0,1)}\big)(t_\ell, \widetilde{X}^M(t_\ell)) \cdot (W(s) - W(t_\ell)) \times \cdot 1_{]t_\ell, t_{\ell+1}]}(s)\, dW(s).$$

Fix $s \in]t_\ell, t_{\ell+1}]$ and recall the above definition of $V(s)$. By the regularity properties of a and σ,

$$\begin{aligned}
&|a(s, X(s)) - a(t_\ell, \widetilde{X}^M(t_\ell)) - V(s)| \\
&\quad \le |a(s, X(s)) - a(t_\ell, X(s))| + |a(t_\ell, X(s)) - a(t_\ell, \widetilde{X}^M(s))| \\
&\qquad + |a(t_\ell, \widetilde{X}^M(s)) - a(t_\ell, \widetilde{X}^M(t_\ell)) \\
&\qquad\qquad - a^{(0,1)}(t_\ell, \widetilde{X}^M(t_\ell)) \cdot (\widetilde{X}^M(s) - \widetilde{X}^M(t_\ell))| \\
&\qquad + \big|a^{(0,1)}(t_\ell, \widetilde{X}^M(t_\ell)) \cdot \big(\widetilde{X}^M(s) - \widetilde{X}^M(t_\ell) \\
&\qquad\qquad - \sigma(t_\ell, \widetilde{X}^M(t_\ell)) \cdot (W(s) - W(t_\ell))\big)\big| \\
&\quad \le c \cdot (1 + |X(s)|) \cdot (s - t_\ell) + c \cdot |X(s) - \widetilde{X}^M(s)| \\
&\qquad + c \cdot |\widetilde{X}^M(s) - \widetilde{X}^M(t_\ell)|^2 \\
&\qquad + c \cdot \big(1 + |\widetilde{X}^M(t_\ell)|\big) \cdot \big((s - t_\ell) + (W(s) - W(t_\ell))^2\big).
\end{aligned}$$

Hence, by property (6) and Lemma 2,

$$\begin{aligned}
&|a(s, X(s)) - a(t_\ell, \widetilde{X}^M(t_\ell)) - V(s)|_q^q \\
&\quad \le c \cdot \big((1 + |X(s)|_q^q + |\widetilde{X}^M(t_\ell)|_q^q) \cdot \Delta_{\max}^q + |X(s) - \widetilde{X}^M(s)|_q^q\big) \\
&\quad \le c \cdot \big(\Delta_{\max}^q + |X(s) - \widetilde{X}^M(s)|_q^q\big)
\end{aligned}$$

Use Lemma 3 and the Hölder inequality to conclude

$$\begin{aligned}
&|A(t) - \widetilde{A}(t)|_q^q \\
&\quad \le c \cdot \int_0^t \sum_{\ell=0}^{n-1} |a(s, X(s)) - a(t_\ell, \widetilde{X}^M(t_\ell)) - V(s)|_q^q \times 1_{]t_\ell, t_{\ell+1}]}(s)\, ds \\
&\qquad + c \cdot |U(t)|_q^q \\
&\quad \le c \cdot \Delta_{\max}^q + c \cdot \int_0^t |X(s) - \widetilde{X}^M(s)|_q^q\, ds. \qquad (11)
\end{aligned}$$

Similarly, for $s \in]t_\ell, t_{\ell+1}]$,

$$\begin{aligned}
&\big|\sigma(s, X(s)) - \sigma(t_\ell, \widetilde{X}^M(t_\ell)) \\
&\qquad - \big(\sigma \cdot \sigma^{(0,1)}\big)(t_\ell, \widetilde{X}^M(t_\ell)) \cdot (W(s) - W(t_\ell))\big|_q^q \\
&\quad \le c \cdot \Delta_{\max}^q + c \cdot |X(s) - \widetilde{X}^M(s)|_q^q.
\end{aligned}$$

Thus, by the Burkholder inequality (assume $q \ge 2$),

$$|\Sigma(t) - \widetilde{\Sigma}(t)|_q^q \le c \cdot \Delta_{\max}^q + c \cdot \int_0^t |X(s) - \widetilde{X}^M(s)|_q^q\, ds. \qquad (12)$$

Combine (11) with (12) to obtain

$$|X(t) - \widetilde{X}^M(t)|_q^q \le c \cdot \Delta_{\max}^q + c \cdot |X(s) - \widetilde{X}^M(s)|_q^q$$

for every $t \in [0,1]$. It remains to apply Gronwall's Lemma to finish the proof. □

Appendix B. Proof of Theorems 6 and 8

We start with a result on approximation of Brownian bridges:

Lemma 4. *Consider a Brownian bridge B on an interval $[S,T]$ and let $\widetilde{B}$ denote the piecewise linear interpolation of B at the points*

$$s_\ell = \ell \cdot (T-S)/(m+1), \quad \ell = 1, \dots, m.$$

Then

$$\int_S^T E\big(B(t) - \widetilde{B}(t)\big)^2 \, dt = \frac{(T-S)^2}{6 \cdot (m+1)}.$$

Proof. Straightforward, using the fact that

$$E\big(B(t) - \widetilde{B}(t)\big)^2 = \frac{(s_{\ell+1} - t) \cdot (t - s_\ell)}{s_{\ell+1} - s_\ell}$$

for $t \in [s_\ell, s_{\ell+1}]$. □

Proof of the lower bound in Theorem 8

Consider an arbitrary sequence of approximations

$$\overline{X}_N = \widehat{X}_{(\psi^{(N)}, \chi^{(N)}, \varphi^{(N)})} \in \mathbb{X}$$

with

$$c(\overline{X}_N) \le N.$$

Let

$$D^{(N)} = D_{\nu_{(\psi^{(N)}, \chi^{(N)}, \varphi^{(N)})}}$$

denote the corresponding underlying Brownian data and define

$$Z_N(t) = W(t) - E(W(t)|D^{(N)}), \quad t \in [0,1].$$

Take a sequence of positive integers k_N such that

$$\lim_{N\to\infty} N^{1/2}/k_N = \lim_{N\to\infty} k_N/N = 0. \qquad (13)$$

Since $k_N = o(N)$ we may assume that $\overline{X}_N$ uses in particular the knots

$$t_\ell^{(N)} = \ell/k_N, \quad \ell = 0, 1, \dots, k_N.$$

Let $\widetilde{X}_N^M$ denote the corresponding time-continuous Milstein scheme, see Appendix A, and put

$$U_\ell^{(N)} = \big(t_\ell^{(N)}, \widetilde{X}_N^M(t_\ell^{(N)})\big).$$

Lemma 5.

$$\liminf_{N\to\infty} N \cdot e^2(\overline{X}_N) \ge \liminf_{N\to\infty} N \cdot \sum_{\ell=0}^{k_N-1} \int_{t_\ell^{(N)}}^{t_{\ell+1}^{(N)}} E\big|\sigma(U_\ell^{(N)}) \cdot Z_N(t)\big|^2 \, dt$$

Proof. Theorem 12 implies

$$E\big\|X - \widetilde{X}_N^M\big\|_2^2 \le c/k_N^2.$$

Since $c/k_N^2 = o(N^{-1})$ by (13), it therefore suffices to analyze the difference $\widetilde{X}_N^M - \overline{X}_N$.

Clearly,

$$E\big|\widetilde{X}_N^M(t) - \overline{X}_N(t)\big|^2 \geq E\big|\widetilde{X}_N^M(t) - E(\widetilde{X}_N^M(t)|D^{(N)})\big|^2$$

Let $t \in [t_\ell^{(N)}, t_{\ell+1}^{(N)}]$. By definition of $\widetilde{X}_N^M$,

$$\begin{aligned}
&\widetilde{X}_N^M(t) - E(\widetilde{X}_N^M(t)\big|D^{(N)}) \\
&\quad = \sigma(U_\ell^{(N)}) \cdot Z_N(t) + 1/2 \cdot (\sigma \cdot \sigma^{(0,1)})(U_\ell^{(N)}) \\
&\qquad \times \big((W(t) - W(t_\ell^{(N)}))^2 - E((W(t) - W(t_\ell^{(N)}))^2|D^{(N)})\big)
\end{aligned}$$

By boundedness of $\sigma^{(0,1)}$ and linear growth of σ,

$$\begin{aligned}
&E\big|(\sigma \cdot \sigma^{(0,1)})(U_\ell^{(N)}) \\
&\qquad \times \big((W(t) - W(t_\ell^{(N)}))^2 - E((W(t) - W(t_\ell^{(N)}))^2|D^{(N)})\big)\big|^2 \\
&\quad \leq c \cdot E\big(\sigma^2(U_\ell^{(N)}) \cdot (W(t) - W(t_\ell^{(N)}))^4\big) \\
&\quad \leq c \cdot \big(1 + E|\widetilde{X}_N^M(t_\ell^{(N)})|^2\big) \\
&\quad \leq c/k_N^2,
\end{aligned}$$

where the last estimate follows from Lemma 2(i).
We conclude that

$$\begin{aligned}
&\big(E\|\widetilde{X}_N^M - \overline{X}_N\|_2^2\big)^{1/2} \\
&\qquad \geq \left(\sum_{\ell=0}^{k_N-1} \int_{t_\ell^{(N)}}^{t_{\ell+1}^{(N)}} E\big|\sigma(U_\ell^{(N)}) \cdot Z_N(t)\big|^2 \, dt\right)^{1/2} - c/k_N,
\end{aligned}$$

and it remains to observe $c/k_N = o(N^{-1/2})$ due to (13). □

The random set of discretization points used by $\overline{X}_N$ is given by

$$\begin{aligned}
\mathcal{M}^{(N)} = \Big\{&\psi_1^{(N)}(x_0), \psi_2^{(N)}\big(x_0, W(\psi^{(N)}(x_0)\big), \\
&\ldots, \psi^{(N)}_{\nu_{(\psi^{(N)},\chi^{(N)},\varphi^{(N)})}}\big(D_{\nu_{(\psi^{(N)},\chi^{(N)},\varphi^{(N)})}-1}\big)\Big\}
\end{aligned}$$

Let

$$m_\ell^{(N)} = \#\big(\mathcal{M}^{(N)} \cap]t_\ell^{(N)}, t_{\ell+1}^{(N)}[\big)$$

denote the random number of points in $]t_\ell^{(N)}, t_{\ell+1}^{(N)}[$ and put

$$A_N = \sum_{\ell=0}^{k_N-1} \frac{\sigma^2(t_\ell^{(N)}, X(t_\ell^{(N)}))}{m_\ell^{(N)} + 1}.$$

Lemma 6.

$$\begin{aligned}
\liminf_{N\to\infty} N \cdot \sum_{\ell=0}^{k_N-1} \int_{t_\ell^{(N)}}^{t_{\ell+1}^{(N)}} E\big|\sigma(U_\ell^{(N)}) \cdot Z_N(t)\big|^2 \, dt \\
\geq \liminf_{N\to\infty} N/(6k_N^2) \cdot E(A_N).
\end{aligned}$$

Proof. Clearly,

$$E\big(|\sigma(U_\ell^{(N)}) \cdot Z_N(t)|^2 \,\big|\, D^{(N)}\big) = \sigma^2(U_\ell^{(N)}) \cdot E\big(Z_N^2(t) \,\big|\, D^{(N)}\big)$$

Conditioned on the data $D^{(N)}$, the discretization $\mathcal{M}^{(N)}$ is fixed and the process Z_N is a Brownian bridge on each of the corresponding subintervals.

Thus

$$\int_{t_\ell^{(N)}}^{t_{\ell+1}^{(N)}} E\big(Z_N^2(t) \mid D^{(N)}\big)\, dt \geq \frac{1}{6k_N^2 \cdot (m_\ell^{(N)}+1)}$$

due to Lemma 4, and consequently

$$\int_{t_\ell^{(N)}}^{t_{\ell+1}^{(N)}} E\big|\sigma(U_\ell^{(N)}) \cdot Z_N(t)\big|^2\, dt \geq 1/(6k_N^2) \cdot E\left(\frac{\sigma^2(U_\ell^{(N)})}{m_\ell^{(N)}+1}\right)$$

By the regularity assumptions on σ,

$$\begin{aligned}&\big|\sigma^2(U_\ell^{(N)}) - \sigma^2(t_\ell^{(N)}, X(t_\ell^{(N)})\big| \\ &\quad \leq c \cdot |\widetilde{X}_N^M(t_\ell^{(N)}) - X(t_\ell^{(N)})| \cdot \big(1 + |\widetilde{X}_N^M(t_\ell^{(N)})| + |X(t_\ell^{(N)})|\big).\end{aligned}$$

Hence, by Theorem 12, Lemma 4 and (6),

$$E\big|\sigma^2(U_\ell^{(N)}) - \sigma^2(t_\ell^{(N)}, X(t_\ell^{(N)})\big|^2 \leq c/k_N,$$

which yields

$$\sum_{\ell=0}^{k_N-1} \int_{t_\ell^{(N)}}^{t_{\ell+1}^{(N)}} E\big|\sigma(U_\ell^{(N)}) \cdot Z_N(t)\big|^2\, dt \geq 1/(6k_N^2) \cdot E(A_N) - c/k_N^2.$$

Employing (13) completes the proof. □

Lemma 7.

$$\liminf_{N\to\infty} N/(6k_N^2) \cdot E(A_N) \geq C^2$$

Proof. By assumption on $\overline{X}_N$ we have

$$\sum_{\ell=0}^{k_N-1} E\big(m_\ell^{(N)}+1\big) = c(\overline{X}_N) \leq N.$$

The Hölder inequality yields

$$\begin{aligned} &N \cdot E(A_N) \\ &\quad \geq \sum_{\ell=0}^{k_N-1} E\left(\frac{\sigma^2(t_\ell^{(N)}, X(t_\ell^{(N)}))}{m_\ell^{(N)}+1}\right) \cdot \sum_{\ell=0}^{k_N-1} E\big(m_\ell^{(N)}+1\big) \\ &\quad \geq \left(\sum_{\ell=0}^{k_N-1}\left(E\left(\frac{\sigma^2(t_\ell^{(N)}, X(t_\ell^{(N)}))}{m_\ell^{(N)}+1}\right)\right)^{1/2} \cdot \Big(E\big(m_\ell^{(N)}+1\big)\Big)^{1/2}\right)^2 \\ &\quad \geq \left(\sum_{\ell=0}^{k_N-1} E\big|\sigma(t_\ell^{(N)}, X(t_\ell^{(N)}))\big|\right)^2 \end{aligned}$$

Thus

$$\begin{aligned} &\liminf_{N\to\infty} N/(6k_N^2) \cdot E(A_N) \\ &\qquad \geq \frac{1}{6} \cdot \left(\lim_{N\to\infty} \frac{1}{k_N} \cdot \sum_{\ell=0}^{k_N-1} E\big|\sigma(t_\ell^{(N)}, X(t_\ell^{(N)}))\big|\right)^2 \\ &\qquad = C^2, \end{aligned}$$

as claimed. □

Combine Lemmas 5-7 to obtain the lower bound in Theorem 8. □

We turn to the analysis of the approximation $\widehat{X}_n$.

Recall the *coarse* discretization

$$t_\ell^{(n)} = \ell/k_n, \quad \ell = 0, \dots, k_n,$$

with

$$\lim_{n\to\infty} n^{1/2}/k_n = \lim_{n\to\infty} k_n/n = 0,$$

that appears in the definition of $\widehat{X}_n$, and let $\widetilde{X}_n^M$ denote the corresponding time-continuous version of the Milstein scheme.

Note that $\widetilde{X}_n^M$ and $\widehat{X}_n$ coincide at the points $t_\ell^{(n)}$.

Proof of Theorem 6(ii). By definition of $\widehat{X}_n$,

$$\begin{aligned} k_n + n/k_n \cdot \sum_{\ell=0}^{k_n-1} E\big|\sigma(t_\ell^{(n)}, \widetilde{X}_n^M(t_\ell^{(n)}))\big| &\le c(\widehat{X}_n) \\ &\le 2k_n + n/k_n \cdot \sum_{\ell=0}^{k_n-1} E\big|\sigma(t_\ell^{(n)}, \widetilde{X}_n^M(t_\ell^{(n)}))\big|. \end{aligned}$$

By the Lipschitz property of σ and Theorem 12,

$$\begin{aligned} E\big|\sigma(t_\ell^{(n)}, \widetilde{X}_n^M(t_\ell^{(n)})) - \sigma(t_\ell^{(n)}, X(t_\ell^{(n)}))\big| &\le c \cdot E\big|\widetilde{X}_n^M(t_\ell^{(n)}) - X(t_\ell^{(n)})\big| \\ &\le c/k_n. \end{aligned}$$

Hence

$$\begin{aligned} &\big|1/n \cdot c(\widehat{X}_n) - \sqrt{6}\cdot C\big| \\ &\quad \le \Big|1/n \cdot c(\widehat{X}_n) - 1/k_n \cdot \sum_{\ell=0}^{k_n-1} E\big|\sigma(t_\ell^{(n)}, \widetilde{X}_n^M(t_\ell^{(n)}))\big|\Big| \\ &\qquad + 1/k_n \cdot \sum_{\ell=0}^{k_n-1} E\big|\sigma(t_\ell^{(n)}, \widetilde{X}_n^M(t_\ell^{(n)})) - \sigma(t_\ell^{(n)}, X(t_\ell^{(n)}))\big| \\ &\qquad + \Big|1/k_n \cdot \sum_{\ell=0}^{k_n-1} E\big|\sigma(t_\ell^{(n)}, X(t_\ell^{(n)}))\big| - \sqrt{6}\cdot C\Big| \\ &\quad \le c \cdot k_n/n + \Big|1/k_n \cdot \sum_{\ell=0}^{k_n-1} E\big|\sigma(t_\ell^{(n)}, X(t_\ell^{(n)}))\big| - \sqrt{6}\cdot C\Big|. \end{aligned}$$

With increasing n the last sum above tends to zero, which finishes the proof. □

We proceed with a comparison of $\widehat{X}_n$ and $\widetilde{X}_n^M$.

Lemma 8.

$$\limsup_{n\to\infty} n \cdot E\|\widehat{X}_n - \widetilde{X}_n^M\|_2^2 \le C/\sqrt{6}.$$

Proof. Let $\mathcal{B}_n$ denote the σ-algebra that is generated by

$$W(t_1^{(n)}), \dots, W(t_{k_n}^{(n)}).$$

Fix $\ell \in \{0, 1, \dots, k_n - 1\}$ and recall that $\widehat{X}_n$ uses

$$\mu_\ell^{(n)} = \big\lceil n/k_n \cdot |\sigma(t_\ell^{(n)}, \widetilde{X}_n^M(t_\ell^{(n)}))|\big\rceil$$

equidistant points

$$\tau_{\ell,j}^{(n)} = t_\ell^{(n)} + j \cdot (k_n \cdot (\mu_\ell^{(n)} + 1))^{-1}, \quad j = 1, \dots, \mu_\ell^{(n)},$$

in the subinterval $]t_\ell^{(n)}, t_{\ell+1}^{(n)}[$.

Note that the adaptive discretization is $\mathcal{B}_n$-measurable and let $\widetilde{W}_n$ denote the piecewise linear interpolation of $W - W(t_\ell^{(n)})$ at these points.

For $t \in [t_\ell^{(n)}, t_{\ell+1}^{(n)}]$ put

$$V_n(t) = 1/2 \cdot \big(\sigma \cdot \sigma^{(0,1)}\big)\big(t_\ell^{(n)}, \widetilde{X}_n^M(t_\ell^{(n)})\big) \times \big((W(t) - W(t_\ell^{(n)}))^2 - (t - t_\ell^{(n)})\big)$$

Then

$$\begin{aligned}&\widetilde{X}_n^M(t) - \widehat{X}_n(t)\\ &\quad = \sigma(t_\ell^{(n)}, \widetilde{X}_n^M(t_\ell^{(n)})) \cdot \big(W(t) - W(t_\ell^{(n)}) - \widetilde{W}_n(t)\big) + V_n(t)\end{aligned}$$

for $t_\ell^{(n)} \le t \le \tau_{\ell,\mu_\ell^{(n)}}^{(n)}$, and

$$\begin{aligned}&|\widetilde{X}_n^M(t) - \widehat{X}_n(t)|\\ &\quad \le \big|\sigma(t_\ell^{(n)}, \widetilde{X}_n^M(t_\ell^{(n)}))\big| \cdot \big|W(t) - W(t_\ell^{(n)}) - \widetilde{W}_n(t)\big|\\ &\qquad + |V_n(t)| + |V_n(t(n)_{\ell+1})|\end{aligned}$$

for $\tau_{\ell,\mu_\ell^{(n)}}^{(n)} < t < t_{\ell+1}^{(n)}$.

Conditioned on $\mathcal{B}_n$ the adaptive discretization is fixed and the process $W - W(t_\ell^{(n)})$ is a Brownian bridge on each of the subintervals $[\tau_{\ell,j}^{(n)}, \tau_{\ell,j+1}^{(n)}]$.

Hence, by Lemma 4,

$$\begin{aligned}&\int_{t_\ell^{(n)}}^{t_{\ell+1}^{(n)}} E\Big(\big|\sigma(t_\ell^{(n)}, \widetilde{X}_n^M(t_\ell^{(n)}))\\ &\qquad \times (W(t) - W(t_\ell^{(n)}) - \widetilde{W}_n(t))\big|^2 \,\big|\, \mathcal{B}_n\Big)\, dt\\ &\quad = \big|\sigma(t_\ell^{(n)}, \widetilde{X}_n^M(t_\ell^{(n)}))\big|^2 \cdot \frac{1}{6k_n^2 \cdot (\mu_\ell^{(n)} + 1)}\\ &\quad \le \frac{\big|\sigma(t_\ell^{(n)}, \widetilde{X}_n^M(t_\ell^{(n)}))\big|}{6k_n \cdot n},\end{aligned}$$

and we obtain

$$\begin{aligned}&\sum_{\ell=0}^{k_n-1} \int_{t_\ell^{(n)}}^{t_{\ell+1}^{(n)}} E\big|\sigma(t_\ell^{(n)}, \widetilde{X}_n^M(t_\ell^{(n)}))\\ &\qquad \times (W(t) - W(t_\ell^{(n)}) - \widetilde{W}_n(t))\big|^2\, dt\\ &\quad \le 1/6 \cdot \frac{1}{k_n} \cdot \sum_{\ell=0}^{k_n-1} E\big|\sigma(t_\ell^{(n)}, \widetilde{X}_n^M(t_\ell^{(n)}))\big|\\ &\quad \le 1/6 \cdot 1/n^2 \cdot c(\widehat{X}_n).\end{aligned}$$

By the regularity conditions on σ and Lemma 2 we have

$$\sup_{t \in [0,1]} E|V_n(t)|^2 \le c/k_n^2.$$

Thus we conclude

$$\big(E\|\widehat{X}_n - \widetilde{X}_n^M\|_2^2\big)^{1/2} \le 1/\sqrt{6} \cdot 1/n \cdot \big(c(\widehat{X}_n)\big)^{1/2} + c/k_n.$$

Now, observe $k_n = o(n^{-1/2})$ and use Theorem 6(ii) to obtain the desired result. □

Proof of the upper bound in Theorem 6(i).

By Theorem 12,

$$\big(c(\widehat{X}_n)\big)^{1/2} \cdot e(\widehat{X}_n)$$
$$\leq \big(c(\widehat{X}_n)\big)^{1/2} \cdot \big((E\|\widetilde{X}_n^M - \widehat{X}_n\|_2^2)^{1/2} + (E\|X - \widetilde{X}_n^M\|_2^2)^{1/2}\big)$$
$$\leq \big(c(\widehat{X}_n)\big)^{1/2} \cdot \big((E\|\widetilde{X}_n^M - \widehat{X}_n\|_2^2)^{1/2} + c/k_n\big).$$

Theorem 6(ii) implies

$$\limsup_{n\to\infty} 1/k_n \cdot \big(c(\widehat{X}_n)\big)^{1/2} \leq c \cdot \limsup_{n\to\infty} n^{1/2}/k_n = 0.$$

Furthermore, by Theorem 6(ii) and Lemma 8,

$$\limsup_{n\to\infty} \big(c(\widehat{X}_n)\big)^{1/2} \cdot \big(E\|\widetilde{X}_n^M - \widehat{X}_n\|_2^2\big)^{1/2} \leq C,$$

which finishes the proof. □

Obviously, the upper bound from Theorem 6(i) implies the upper bound from Theorem 8, while the lower bound from Theorem 8 yields the lower bound from Theorem 6(i).

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